The closure property

Adding two whole numbers always gives a whole number, but subtracting them sometimes does not. When does an operation keep every answer inside a set of numbers?

WORDS TO KNOW

Set: a collection of numbers, such as the whole numbers 0, 1, 2, …

Member: a number that belongs to the set. $a \in S$ means a is a member of S, and $a \notin S$ means it is not.

Closed under an operation: doing that operation on any two members, even a member with itself, always gives a member.

Counterexample: one case that shows a claim about every case is false.

$\mathbb{Q}$**:** the rational numbers.

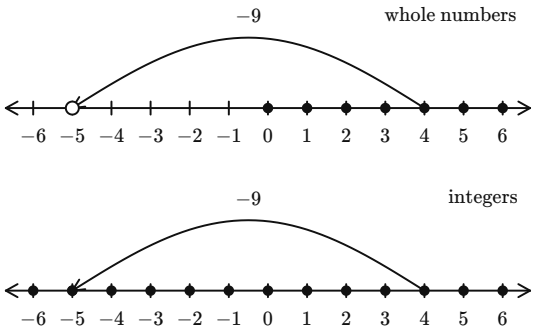
KEY IDEAS

1. Closed means the answer never leaves the set

Adding moves right, and the whole numbers go on forever to the right. Subtracting moves left, and the whole numbers stop at 0. In the picture, 4 − 9 runs off the end of the whole numbers but lands on an integer.

To show a set **is** closed, give a reason that covers every pair, often with letters. Examples are not enough, unless the set is small enough to list every pair.

To show a set is **not** closed, one counterexample is enough.



2. The closure of the familiar sets

Division always means division by a nonzero member.

Set	Closed under	Not closed under
natural numbers	$+$, $\times$	$-$, $\div$
whole numbers	$+$, $\times$	$-$, $\div$
integers	$+$, $-$, $\times$	$\div$
rational numbers	all four	none
irrational numbers	none	all four
real numbers	all four	none

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3. A rational number with an irrational one

If r is rational and x is irrational, then $r + x$, $r - x$ and $x - r$ are irrational, and so are $r \times x$ and $x \div r$ when $r \neq 0$.

WATCH OUT

- **Name the operation.** "The integers are closed" means nothing alone: they are closed under subtraction, not under division.
- **Not closed does not mean every answer escapes.** $\pi + \pi = 2\pi$ is irrational, yet the irrational numbers are not closed under addition, because some sums are rational.
- **The product rule needs** $r \neq 0$**,** since $0 \times \pi = 0$ is rational.

WORKED EXAMPLES

Example 1. Is the set $\{-1, 1\}$ closed under multiplication? Under subtraction?

1. There are only four ordered pairs. List every product.
$$1 \times 1 = 1 \qquad 1 \times (-1) = -1 \qquad (-1) \times 1 = -1 \qquad (-1) \times (-1) = 1$$
2. Every product is −1 or 1, so the set is closed under multiplication.
3. For subtraction, one pair is enough: (−1) − 1 = −2, and $-2 \notin \{-1, 1\}$.

Answer: closed under multiplication, not closed under subtraction.

Example 2. Are the multiples of 3 (the integers $3n$, with n an integer) closed under multiplication? Under division?

1. Take any two members, $3m$ and $3n$. Then $3m \times 3n = 9mn = 3(3mn)$.
2. $3mn$ is an integer, as the integers are closed under multiplication. So every product is a multiple of 3.
3. For division, one counterexample: $6 \div 9 = 2/3$, which is not an integer.

Answer: closed under multiplication, not closed under division.

Example 3. Let $p = 1 + 4\sqrt{3}$ and $q = 1 - 4\sqrt{3}$, both irrational. Is $p + q$, $p - q$ or $p \times q$ rational?

1. Sum: the $\sqrt{3}$ parts cancel, so $p + q = 2$, rational.
2. Difference: the 1s cancel, so $p - q = 8\sqrt{3}$, irrational (a nonzero rational times $\sqrt{3}$).
3. Product: use the distributive property and $\sqrt{3} \times \sqrt{3} = 3$.
$$p \times q = 1 - 4\sqrt{3} + 4\sqrt{3} - 16 \times 3 = -47$$

Answer: $p + q = 2$ and $p \times q = -47$ are rational; $p - q = 8\sqrt{3}$ is irrational. Two irrational numbers can give either kind of answer.

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