Algebra 1 Unit 1 test: Real numbers, expressions and properties (answer key)

13 questions, 45 points

ANSWERS AT A GLANCE

1. $\{6, 7, 8, 9, 10\}$, with 5 elements
2. $S = \{3x + 30,\ 3x + 24\}$ for every real number x.
3. $3x + 72$
4. (a) (−3, −3), (−1, −1), (1, 1), (3, 3), (6, 6): 5 pairs. (b) those five, and (−3, 3), (3, −3), (−1, 1), (1, −1): 9 pairs.
5. (a) $\{-3, -1, 0, 1, 3\}$ (b) $\{-1, \frac{1}{3}, 1, 3\}$ (c) own multiplicative inverse: $\{-1, 1\}$; own additive inverse: $\{0\}$
6. $x = \dfrac{\sqrt{13} + 48}{11}$, irrational
7. $x = -\sqrt{5}$ only.
8. (a) Liters: the water drained in t minutes. (b) 6210 L (c) 6 hours
9. (a) $(x + 5)^2 + 4 = (x + 5)(x + 5) + 4 = x^2 + 5x + 5x + 25 + 4 = x^2 + 10x + 29$. (b) Isaac is wrong: a square plus 4 is at least 4, so the expression is never negative; its smallest value is 4, at $x = -5$. (c) $x = -2$ and $x = -8$
10. (a) $e = \dfrac{1}{2}$ (b) $b = \dfrac{1}{4a}$, for $a \neq 0$ (c) 0, since $0 * b = 0$ for every b, never $\dfrac{1}{2}$.
11. (a) 1800 L (b) about 1800 L, two significant figures, limited by the 18 mm and the 8.0 m (c) 8 barrels
12. About 3.2×10^6 turns
13. (a) The sum in parentheses is the cost of one lunch; the 2 counts the two lunches; 40 is the one delivery fee. (b) $42g + 220$: 42 dollars is what one more guest at both lunches adds, and 220 dollars is both setup charges plus delivery, which does not depend on g. (c) 2068 dollars

WORKED SOLUTIONS

1. Write as a roster the set of integers n for which $3(n + 4) - 2n$ is greater than 17 and less than 23. How many elements does it have? (3 points)

ANSWER

$\{6, 7, 8, 9, 10\}$, with 5 elements

SOLUTION

Simplify first. Distribute, then combine like terms:

$$3(n + 4) - 2n = 3n + 12 - 2n = n + 12$$

So we need $n + 12$ to be greater than 17 and less than 23. That happens exactly when n is greater than 5 and less than 11. The integers strictly between 5 and 11 give

$$\{6, 7, 8, 9, 10\}$$

with 5 elements.

SCORING

1 point for simplifying the expression to $n + 12$; 1 point for the bounds 5 and 11 on n; 1 point for the roster with the ends left out.

2. Let $S = \{5(x + 6) - 2x,\ \ 3(x + 6) + 12,\ \ 3x + 24\}$. Show that for every real number x the set S has exactly two elements, and write S with each element listed once. (3 points)

ANSWER

$S = \{3x + 30,\ 3x + 24\}$ for every real number x.

SOLUTION

Simplify each listed expression.

First: $5(x + 6) - 2x = 5x + 30 - 2x = 3x + 30$, by the distributive property and then combining like terms.

Second: $3(x + 6) + 12 = 3x + 18 + 12 = 3x + 30$.

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Third: $3x + 24$.

The first two are the same expression, so for every x they are the same number, and the repeat collapses. The third differs from them by 6 for every x, since $(3x + 24) - (3x + 30) = -6$, which is never 0. So for every real number x the set has exactly two different elements:

$$S = \{3x + 30,\ 3x + 24\}.$$

SCORING
1 point for simplifying the first two expressions to $3x + 30$. 1 point for showing the third always differs from them (the difference is the constant −6). 1 point for concluding that S has exactly two elements for every x, written once each.

3. Simplify $-9(x - 9) - (9 - 12x)$. Show each step. (3 points)

ANSWER
$3x + 72$

SOLUTION
Distribute −9: $-9(x - 9) = -9x + 81$.

Subtracting the part in parentheses is adding its opposite: $-(9 - 12x) = -9 + 12x$.

So the expression is $-9x + 81 - 9 + 12x$. Bring like terms together (commutative and associative properties of addition, each term keeping its sign): $(-9x + 12x) + (81 - 9)$.

Combine like terms (the distributive property read right to left): $(-9 + 12)x + (81 - 9) = 3x + 72$.

SCORING
1 point for $-9x + 81$; 1 point for $-9 + 12x$ (the sign of every term in the second set of parentheses changed); 1 point for $3x + 72$.

4. Let $S = \{-3, -1, 1, 3, 6\}$. Here a and b are elements of S, and they may be equal. (3 points)

(a) List every ordered pair (a, b) for which $a - b = b - a$.

(b) List every ordered pair (a, b) for which $a \div b = b \div a$.

Explain how you found each list without trying all 25 pairs.

ANSWER
(a) (−3, −3), (−1, −1), (1, 1), (3, 3), (6, 6): 5 pairs. (b) those five, and (−3, 3), (3, −3), (−1, 1), (1, −1): 9 pairs.

SOLUTION
(a) $a - b = b - a$ means $a - b$ is its own opposite, so $a - b = 0$ and $a = b$. The pairs are the five with equal entries: (−3, −3), (−1, −1), (1, 1), (3, 3), (6, 6).

(b) No element of S is 0, so both quotients exist. $\frac{a}{b} = \frac{b}{a}$ exactly when $a \cdot a = b \cdot b$ (multiply both sides by ab), that is, when $b = a$ or $b = -a$. The equal pairs from (a) all work, and so do the pairs of opposites that lie in S: (−3, 3), (3, −3), (−1, 1), (1, −1). The number 6 has no opposite in S. That makes 9 pairs.

Division is not commutative, yet it gives the same answer for these 9 of the 25 pairs: agreement for some numbers is not a law.

SCORING
1 point for the five pairs in (a) with the reason $a = b$; 1 point for the condition $b = a$ or $b = -a$ in (b); 1 point for the complete list of 9 pairs in (b).

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5. Let $S = \{-3, -1, 0, \frac{1}{3}, 1, 3, 4\}$. Write each set as a roster. (3 points)

(a) The elements of S whose additive inverse is also an element of S.

(b) The elements of S whose multiplicative inverse is also an element of S.

(c) The elements of S that are their own multiplicative inverse, and the elements of S that are their own additive inverse.

ANSWER

(a) $\{-3, -1, 0, 1, 3\}$ (b) $\{-1, \frac{1}{3}, 1, 3\}$ (c) own multiplicative inverse: $\{-1, 1\}$; own additive inverse: $\{0\}$

SOLUTION

(a) The opposites are $3, 1, 0, -\frac{1}{3}, -1, -3, -4$. Of these, $-\frac{1}{3}$ and -4 are not in S, so $\frac{1}{3}$ and 4 drop out: $\{-3, -1, 0, 1, 3\}$.

(b) 0 has no reciprocal. The others have reciprocals $-\frac{1}{3}, -1, 3, 1, \frac{1}{3}, \frac{1}{4}$. The reciprocals $-\frac{1}{3}$ (of -3) and $\frac{1}{4}$ (of 4) are not in S: $\{-1, \frac{1}{3}, 1, 3\}$.

(c) $a \cdot a = 1$ only for $a = 1$ and $a = -1$, both in S: $\{-1, 1\}$. $a + a = 0$ only for $a = 0$: $\{0\}$.

SCORING

1 point for each part, with every element right; in (b), no credit if 0 is listed.

6. Solve $8(x - 6) = -3x + \sqrt{13}$. Give x in simplest form, then decide whether x is rational or irrational, naming the closure facts you use. (3 points)

ANSWER

$x = \frac{\sqrt{13} + 48}{11}$, irrational

SOLUTION

Distribute: $8x - 48 = -3x + \sqrt{13}$.

Collect the x terms on the left: $11x - 48 = \sqrt{13}$.

Undo the constant: $11x = \sqrt{13} + 48$.

Divide by 11: $x = \frac{\sqrt{13} + 48}{11}$.

$\sqrt{13}$ is irrational because 13 is not a perfect square. Suppose x were rational. Then $\sqrt{13} = 11x - 48$ would be rational, because the rational numbers are closed under multiplication, addition and subtraction. It is not, so x is irrational.

SCORING

1 point for distributing and collecting to $11x - 48 = \sqrt{13}$; 1 point for $x = \frac{\sqrt{13} + 48}{11}$; 1 point for "irrational" with a reason that uses closure of the rational numbers (or the rule for a rational number with an irrational one).

7. Find every irrational number x for which both $x + \sqrt{5}$ and $x \cdot \sqrt{5}$ are rational. Explain why there are no others. (3 points)

ANSWER

$x = -\sqrt{5}$ only.

SOLUTION

Suppose $x + \sqrt{5} = r$ with r rational. Then $x = r - \sqrt{5}$, and by the distributive property

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$$x \cdot \sqrt{5} = r\sqrt{5} - 5$$

If this is rational, then $r\sqrt{5}$ is rational, because adding 5 keeps it rational (closure of the rational numbers under addition). But a nonzero rational number times $\sqrt{5}$ is irrational, so r must be 0. Then $x = -\sqrt{5}$. Check: $-\sqrt{5}$ is irrational, $-\sqrt{5} + \sqrt{5} = 0$ and $-\sqrt{5} \cdot \sqrt{5} = -5$, both rational. So $x = -\sqrt{5}$ is the only such number.

SCORING

1 point for writing $x = r - \sqrt{5}$ with r rational; 1 point for showing $x\sqrt{5} = r\sqrt{5} - 5$ forces $r = 0$ by closure; 1 point for $x = -\sqrt{5}$ with the check.

8. A tank holds 8280 liters of water. It drains at 23 liters per minute, and after t minutes the water left in the tank is $8280 - 23t$ liters. (4 points)

(a) Give the unit of the term $23t$ and say what it measures.

(b) How many liters are left after 1 hour 30 minutes?

(c) How many hours does it take the tank to empty?

ANSWER

(a) Liters: the water drained in t minutes. (b) 6210 L (c) 6 hours

SOLUTION

(a) $\dfrac{23\text{ L}}{1\text{ min}} \times t$ min: minutes cancel, so $23t$ is in liters. It is the amount of water that has drained out in the first t minutes, which is why it is subtracted from the 8280 liters the tank started with.

(b) The formula wants t in minutes: 1 hour 30 minutes is 90 minutes. Then $23 \times 90 = 2070$ and $8280 - 2070 = 6210$ liters.

(c) The tank is empty when $8280 - 23t = 0$, so $t = 8280 \div 23$ minutes. In hours:

$$8280\text{ L} \times \frac{1\text{ min}}{23\text{ L}} \times \frac{1\text{ h}}{60\text{ min}} = 6\text{ h}$$

SCORING

1 point for (a): liters, the water drained so far. 1 point for (b) 6210 L, using $t = 90$. 2 points for (c): 1 for a correct set-up turning liters into minutes and minutes into hours, 1 for 6 hours.

9. Isaac looks at $x^2 + 10x + 29$ and says: "When x is a large negative number, the term $10x$ is very negative, so for some x the whole expression must be negative." (4 points)

(a) Use the distributive property and combine like terms to show that $(x + 5)^2 + 4 = x^2 + 10x + 29$ for every real number x.

(b) Use the form $(x + 5)^2 + 4$ to decide whether Isaac is right. What is the smallest value the expression can take?

(c) Find every x for which $x^2 + 10x + 29 = 13$.

ANSWER

(a) $(x + 5)^2 + 4 = (x + 5)(x + 5) + 4 = x^2 + 5x + 5x + 25 + 4 = x^2 + 10x + 29$. (b) Isaac is wrong: a square plus 4 is at least 4, so the expression is never negative; its smallest value is 4, at $x = -5$. (c) $x = -2$ and $x = -8$

SOLUTION

(a) $(x + 5)^2 = (x + 5)(x + 5)$. Distribute: $x(x + 5) + 5(x + 5) = x^2 + 5x + 5x + 25$, and the like terms $5x + 5x$ combine to $10x$, giving $x^2 + 10x + 25$. Adding 4 gives $x^2 + 10x + 29$.

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(b) The two forms are equal for every x, so read the second: a square plus 4. The square is never negative, so the sum is at least 4 and is never negative. Isaac is wrong: when x is a large negative number, $10x$ is very negative but x^2 is larger still, and the form $x^2 + 10x + 29$ hides this. The smallest value is 4, when $x + 5 = 0$, at $x = -5$.

(c) Use the form from (a): $(x + 5)^2 + 4 = 13$, so $(x + 5)^2 = 9$. Then $x + 5 = 3$ or $x + 5 = -3$: $x = -2$ or $x = -8$. Check: $(-2)^2 + 10(-2) + 29 = 4 - 20 + 29 = 13$.

SCORING

1 point for a correct expansion with like terms combined; 1 point for saying Isaac is wrong because a square plus 4 is at least 4; 1 point for the smallest value 4 at $x = -5$; 1 point for both $x = -2$ and $x = -8$.

10. A new operation $*$ is defined on the real numbers by $a * b = 2ab$. (4 points)

(a) Find the identity for $*$: a number e with $a * e = a$ and $e * a = a$ for every real number a.

(b) For a real number a, find its inverse under $*$: a number b with $a * b = e$, where e is your answer to (a).

(c) Which real number has no inverse under $*$? Explain.

ANSWER

(a) $e = \dfrac{1}{2}$ (b) $b = \dfrac{1}{4a}$, for $a \neq 0$ (c) 0, since $0 * b = 0$ for every b, never $\dfrac{1}{2}$.

SOLUTION

(a) $a * e = 2ae$ must equal a for every a. Taking $a = 1$ gives $2e = 1$, so $e = \dfrac{1}{2}$, and then $2a \cdot \dfrac{1}{2} = a$ for every a. Since multiplication is commutative, $e * a = 2ea = 2ae = a$ as well. So $e = \dfrac{1}{2}$.

(b) An inverse brings a to the identity, which here is $\dfrac{1}{2}$, not 1. Solve $2ab = \dfrac{1}{2}$: multiply both sides by the reciprocal of $2a$, which needs $a \neq 0$, to get $b = \dfrac{1}{2} \cdot \dfrac{1}{2a} = \dfrac{1}{4a}$. Check: $a * \dfrac{1}{4a} = 2a \cdot \dfrac{1}{4a} = \dfrac{2a}{4a} = \dfrac{1}{2}$.

(c) $0 * b = 2 \cdot 0 \cdot b = 0$ for every b, and $0 \neq \dfrac{1}{2}$, so no b works: 0 has no inverse under $*$.

SCORING

1 point for $e = \dfrac{1}{2}$ with both orders checked (or the commutative property cited); 1 point for setting $a * b$ equal to the identity $\dfrac{1}{2}$ (setting it equal to 1 earns 0 here); 1 point for $b = \dfrac{1}{4a}$; 1 point for 0 with the reason $0 * b = 0$.

11. A storm drops 18 mm of rain on a flat garage roof 12.5 m long and 8.0 m wide. All of the rain runs into barrels that hold 250 liters each. (5 points)

(a) Find the volume of rain on the roof in liters. Show a chain whose units cancel to liters.

(b) Give the volume to a sensible accuracy, and say which measurement limits it.

(c) How many barrels are needed to catch all of the rain?

ANSWER

(a) 1800 L (b) about 1800 L, two significant figures, limited by the 18 mm and the 8.0 m (c) 8 barrels

SOLUTION

(a) The roof's area is $12.5\ \text{m} \times 8.0\ \text{m} = 100\ \text{m}^2$. The depth must be in meters before it multiplies the area: $18\ \text{mm} \times \dfrac{1\ \text{m}}{1000\ \text{mm}} = 0.018\ \text{m}$. Then

$$100\ \text{m}^2 \times 0.018\ \text{m} \times \frac{1000\ \text{L}}{1\ \text{m}^3} = 1800\ \text{L}$$

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since $\mathrm{m}^2 \times \mathrm{m} = \mathrm{m}^3$, which cancels against the cubic meters underneath.

(b) The 18 mm and the 8.0 m each have two significant figures (the 12.5 has three), so the answer keeps two: about 1800 liters.

(c) $1800\ \mathrm{L} \times \dfrac{1\ \text{barrel}}{250\ \mathrm{L}} = 7.2$ barrels. Seven barrels would overflow and a part of a barrel cannot be used, so the situation rounds up: 8 barrels.

SCORING

1 point for the area 100 m^2; 1 point for converting the depth to 0.018 m (or the area to square millimeters correctly); 1 point for 1800 L; 1 point for two significant figures with the limiting measurement named; 1 point for 8 barrels, rounding up because 7 is too few.

12. Hannah plans a bicycle ride of 5,000 km across a country. The bicycle's wheels are about 50 cm across. Estimate how many times each wheel turns on the ride. Show the conversion factors with the units canceling, and give the answer in scientific notation to two significant figures. (3 points)

ANSWER

About 3.2×10^6 turns

SOLUTION

One turn moves the bicycle one circumference of the wheel: $\pi \times 50 = 50\pi$ cm.

The distance in centimeters:

$$5000\ \mathrm{km} \times \frac{1000\ \mathrm{m}}{1\ \mathrm{km}} \times \frac{100\ \mathrm{cm}}{1\ \mathrm{m}} = 5000 \times 10^5\ \mathrm{cm}$$

Then

$$\frac{5000 \times 10^5\ \mathrm{cm}}{50\pi\ \mathrm{cm\ per\ turn}} \approx 3.2 \times 10^6\ \mathrm{turns}$$

SCORING

1 point for the distance converted to centimeters (or the circumference to kilometers) with units canceling; 1 point for using the circumference 50π cm as the distance per turn; 1 point for 3.2×10^6 turns.

13. Hannah caters two identical lunches, one on Saturday and one on Sunday, each for g guests. Each lunch costs 21 dollars a guest plus a setup charge of 90 dollars, and there is one delivery fee of 40 dollars for the weekend. The total charge is

$$2(21g + 90) + 40$$

dollars. (4 points)

(a) Say what the sum in parentheses, $21g + 90$, the factor 2 and the term 40 stand for.

(b) Use the distributive property and combine like terms to write the total with one term in g and one constant. Say what the new coefficient of g and the new constant mean.

(c) Find the total charge when each lunch has 44 guests.

ANSWER

(a) The sum in parentheses is the cost of one lunch; the 2 counts the two lunches; 40 is the one delivery fee. (b) $42g + 220$: 42 dollars is what one more guest at both lunches adds, and 220 dollars is both setup charges plus delivery, which does not depend on g. (c) 2068 dollars

SOLUTION

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(a) The last step is adding, so the total is a sum of two terms, $2(21g + 90)$ and 40. In the first term the sum in parentheses is one lunch: 21 dollars for each of g guests plus the 90-dollar setup. The factor 2 makes it two lunches. The term 40 is the delivery fee, charged once and so outside the parentheses.

(b) Distribute: $2 \cdot 21g + 2 \cdot 90 + 40 = 42g + 180 + 40$. Combine the constants: $42g + 220$. The coefficient 42 is 21 dollars twice: what one more guest at both lunches adds. The constant 220 is the part that does not change with g: two setups, 180 dollars, and the delivery, 40 dollars.

(c) $42 \times 44 + 220 = 2068$ dollars. (Check from the first form: $2(21 \times 44 + 90) + 40 = 2068$.)

SCORING

1 point for the meaning of all three parts; 1 point for $42g + 220$; 1 point for the meanings of 42 and 220; 1 point for 2068 dollars.

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