Algebra 1 Unit 1 worksheet: Closure (answer key)

ANSWERS AT A GLANCE

1. (a) −4, the integers (b) $\frac{7}{3}$, the rational numbers (c) 0, the whole numbers (d) 21, the natural numbers
2. She tested whether subtraction is commutative, not closure. Both answers are integers. The integers are closed under subtraction.
3. (a) Yes. (b) No: for example 12 − 11 = 1, which is not in G.
4. (a) No: 6 + 6 = 12. (b) Yes: $(10m + 6)(10n + 6) = 10(10mn + 6m + 6n + 3) + 6$.
5. (a) $x^2 = 5$, rational (b) $x^3 = 5\sqrt{5}$, irrational (c) $x^4 = 25$, rational (d) $x^3 - x = 4\sqrt{5}$, irrational. Part (a) shows the irrational numbers are not closed under multiplication.
6. Answers vary. For example: (a) $\sqrt{2}$ and $3\sqrt{2}$ (b) π and $6 - \pi$ (c) $\sqrt{5}$ and $3\sqrt{5}$
7. (a) If $\frac{1}{x}$ were a rational number s, then $x = 1 \div s$ would be rational, which it is not. (b) Irrational.
8. Closed under addition only. Not closed under subtraction, multiplication or division.

WORKED SOLUTIONS

1. Each answer below comes from two natural numbers. Work it out, then name the smallest of these sets that contains it: the natural numbers, the whole numbers, the integers, the rational numbers.

(a) 3 − 7 (b) 7 ÷ 3 (c) 3 − 3 (d) 7 × 3

ANSWER

(a) −4, the integers (b) $\frac{7}{3}$, the rational numbers (c) 0, the whole numbers (d) 21, the natural numbers

SOLUTION

(a) 3 − 7 = −4. It is less than 0, so it is not a whole number; it is an integer.

(b) $7 \div 3 = \frac{7}{3}$. Since 7 is not a multiple of 3, this is no integer; it is a rational number.

(c) 3 − 3 = 0. Zero is a whole number but not a natural number.

(d) 7 × 3 = 21, a natural number.

Parts (a), (b) and (c) each start with two natural numbers and land outside them, so each is a counterexample: the natural numbers are not closed under subtraction or division.

2. Hannah writes: "7 − 3 = 4 but 3 − 7 = −4. The answers are different, so the integers are not closed under subtraction."

What is wrong with her reasoning? Are the integers closed under subtraction?

ANSWER

She tested whether subtraction is commutative, not closure. Both answers are integers. The integers are closed under subtraction.

SOLUTION

Closure asks only where each answer lands. Here 7 − 3 = 4 and 3 − 7 = −4 are both integers, so neither answer leaves the set. That the two answers differ shows that subtraction is not commutative, which is a different property.

The integers are closed under subtraction: for integers a and b, $a - b = a + (-b)$, the opposite $-b$ is an integer, and the sum of two integers is an integer.

3. Let G be the set of integers greater than 10: 11, 12, 13, ...

(a) Is G closed under addition? Explain.

(b) Ryan works out 30 − 12 = 18, 50 − 25 = 25 and 41 − 11 = 30. Every answer is in G, so he says G is closed under subtraction. Is he right? Explain.

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ANSWER

(a) Yes. (b) No: for example 12 − 11 = 1, which is not in G.

SOLUTION

(a) Yes. If $a > 10$ and $b > 10$, then $a + b > 20$, so $a + b > 10$; and the sum of two integers is an integer. So $a + b$ is in G for every pair.

(b) No. Examples that land inside cannot show closure, but one pair that lands outside is enough to show it fails: 12 − 11 = 1, and 1 is not greater than 10. (Also 11 − 11 = 0.)

4. Let D be the set of natural numbers whose last digit is 6: 6, 16, 26, 36, … Each member can be written as $10m + 6$ with m a whole number.

(a) Is D closed under addition?

(b) Is D closed under multiplication?

For each, give a counterexample or an argument with letters that covers every pair.

ANSWER

(a) No: 6 + 6 = 12. (b) Yes: $(10m + 6)(10n + 6) = 10(10mn + 6m + 6n + 3) + 6$.

SOLUTION

(a) Not closed. 6 + 6 = 12, whose last digit is 2.

(b) Closed. Take two members, $10m + 6$ and $10n + 6$, with m and n whole numbers. By the distributive property,

$$\begin{aligned}(10m + 6)(10n + 6) &= 100mn + 60m + 60n + 36 \\ &= 10(10mn + 6m + 6n + 3) + 6\end{aligned}$$

The number $10mn + 6m + 6n + 3$ is a whole number, because the whole numbers are closed under addition and multiplication. So the product is 10 times a whole number plus 6, which ends in 6. For instance 16 × 26 = 416.

5. Let $x = \sqrt{5}$, an irrational number. Write each number in simplest form and say whether it is rational or irrational.

(a) x^2 (b) x^3 (c) x^4 (d) $x^3 - x$

Then say what part (a) shows about the irrational numbers under multiplication.

ANSWER

(a) $x^2 = 5$, rational (b) $x^3 = 5\sqrt{5}$, irrational (c) $x^4 = 25$, rational (d) $x^3 - x = 4\sqrt{5}$, irrational. Part (a) shows the irrational numbers are not closed under multiplication.

SOLUTION

$\sqrt{5}$ is irrational because 5 is not a perfect square.

(a) $x^2 = \sqrt{5} \times \sqrt{5} = 5$, rational.

(b) $x^3 = x^2 \times x = 5\sqrt{5}$, a nonzero rational number times an irrational one, so irrational.

(c) $x^4 = x^2 \times x^2 = 5 \times 5 = 25$, rational.

(d) By the distributive property, $x^3 - x = 5\sqrt{5} - \sqrt{5} = (5 - 1)\sqrt{5} = 4\sqrt{5}$. Since 4 is a nonzero rational number, this is irrational.

In part (a), two irrational numbers (x and x itself) multiply to the rational number 5, so the irrational numbers are not closed under multiplication.

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6. Give an example of each, or explain why there is none.

(a) Two irrational numbers whose product is 6.

(b) Two irrational numbers whose sum is 6.

(c) Two irrational numbers whose product is rational but whose sum is irrational.

ANSWER

Answers vary. For example: (a) $\sqrt{2}$ and $3\sqrt{2}$ (b) π and $6 - \pi$ (c) $\sqrt{5}$ and $3\sqrt{5}$

SOLUTION

Each part has many answers; these are examples.

(a) $\sqrt{2} \times 3\sqrt{2} = 3 \times 2 = 6$. Both are irrational: $\sqrt{2}$ is, and $3\sqrt{2}$ is a nonzero rational times an irrational number.

(b) $\pi + (6 - \pi) = 6$. The number $6 - \pi$ is a rational number minus an irrational one, so it is irrational.

(c) $\sqrt{5} \times 3\sqrt{5} = 15$, which is rational, while $\sqrt{5} + 3\sqrt{5} = 4\sqrt{5}$, a nonzero rational times $\sqrt{5}$, which is irrational.

Parts (a) and (b) show again that the irrational numbers are not closed under multiplication or addition.

7. (a) Show that if x is irrational, then $\dfrac{1}{x}$ is irrational. (Note that $x \neq 0$, since 0 is rational.)

(b) Use part (a) to decide whether $\dfrac{1}{\pi - 3}$ is rational or irrational.

ANSWER

(a) If $\dfrac{1}{x}$ were a rational number s, then $x = 1 \div s$ would be rational, which it is not. (b) Irrational.

SOLUTION

(a) Suppose $\dfrac{1}{x}$ were a rational number s. Then $s \neq 0$, because 1 divided by a number is never 0. From $\dfrac{1}{x} = s$ we get $x = \dfrac{1}{s} = 1 \div s$. The rational numbers are closed under division by a nonzero member, so $1 \div s$ is rational, and x would be rational. It is not, so the supposition was false: $\dfrac{1}{x}$ is irrational.

(b) $\pi - 3$ is an irrational number minus a rational one, so it is irrational. By part (a), its reciprocal $\dfrac{1}{\pi - 3}$ is irrational.

8. Decide whether the negative rational numbers (the rational numbers less than 0) are closed under each operation: addition, subtraction, multiplication, and division by a nonzero member. For each, give a reason that covers every pair or a counterexample.

ANSWER

Closed under addition only. Not closed under subtraction, multiplication or division.

SOLUTION

Addition: closed. The sum of two rational numbers is rational, by closure of the rational numbers, and two numbers below 0 add to a number below 0.

Subtraction: not closed. −1 − (−3) = 2, which is positive.

Multiplication: not closed. $\left(-\dfrac{1}{2}\right)(-4) = 2$, which is positive.

Division: not closed. $(-1) \div (-2) = \dfrac{1}{2}$, which is positive.

In each failing case the answer is still rational; it only leaves the negative numbers.

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