Algebra 1 Unit 1 worksheet: Estimating a huge quantity from small facts (answer key)

ANSWERS AT A GLANCE

1. About 2.1×10^6 sheets a year, order of magnitude 10^6.
2. (a) $H = 750$ (b) 25 times, the square of 5
3. (a) 4 factors (b) within a factor of 625
4. (a) Lesson's sense: 10^4, 10^2, 10^{-2}, 10^8. Second sense: 10^4, 10^3, 10^{-1}, 10^9. (b) 3.2 is more than $\sqrt{10} \approx 3.16$, so on the power-of-ten scale 3.2×10^8 is closer to 10^9 than to 10^8.
5. (a) 24 texts per student (b) About 28,800 texts, with bounds 3,600 and 230,400 texts.
6. (a) Fewest 2 hours, most 18 hours (b) 6 hours
7. (a) dogs $\approx$ people $\times \dfrac{\text{1 household}}{\text{people per household}} \times \dfrac{\text{dogs}}{\text{household}}$ (b) low about 6.7×10^4, high 6×10^5 (c) 9 times (d) the share of households with a dog
8. (a) 3,750 pizzas (b) 4 times (c) one person in 2
9. One reasonable chain: a sheet is about 0.1 mm thick (surely between 0.05 and 0.2 mm), giving about 4×10^{12} sheets, with bounds about 2×10^{12} and 8×10^{12}. Other stated assumptions give other numbers of the same order.

WORKED SOLUTIONS

1. Hannah's school has 1300 students. Assume each student uses about 9 sheets of paper on a school day, and that a school year has 180 school days. Estimate how many sheets of paper the school uses in a school year. Write the chain with its units, give the answer in scientific notation to two significant figures, and give its order of magnitude.

   ANSWER
   About 2.1×10^6 sheets a year, order of magnitude 10^6.

   SOLUTION

   $$1300 \text{ students} \times \frac{9 \text{ sheets per day}}{1 \text{ student}} \times \frac{180 \text{ days}}{1 \text{ year}}$$

   Students cancel, then days cancel, leaving sheets per year. First 1300 × 9 = 11700 sheets a day, then 11700 × 180 gives 2,106,000 sheets a year.

   To two significant figures that is 2.1×10^6 sheets a year, order of magnitude 10^6. The inputs are guesses, so the later digits mean nothing.

2. Hannah is sure that a quantity is more than 30 and less than some high guess H, and works out that the middle of this range on the power-of-ten scale is 150.

   (a) Find H.

   (b) How many times the low end is the high end? Explain how that number is related to how many times the low end the middle is.

   ANSWER
   (a) $H = 750$ (b) 25 times, the square of 5

   SOLUTION

   (a) The middle M is the same number of times above the low end as it is below the high end: $\dfrac{M}{30} = \dfrac{H}{M}$.
   Here $\dfrac{150}{30} = 5$, so $H = 5 \times 150 = 750$. (Equivalently $H = \dfrac{M^2}{30}$.)
   (b) 750 ÷ 30 = 25. The middle is 5 times the low end and the high end is another 5 times the middle, so the high end is 5 × 5 = 25 times the low end.

3. Hannah can guess each factor of a chain to within a factor of 5 of its true value, and wants a product that is sure to be within a factor of 1000 of the true quantity.

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(a) What is the greatest number of factors the chain can have?

(b) With that many factors, within what factor of the truth is the product sure to be?

ANSWER

(a) 4 factors (b) within a factor of 625

SOLUTION

With k factors, each within a factor 5, the product is within a factor 5^k of the true product. So we need the largest k with $5^k \le 1000$.

$5^4 = 625 \le 1000$, but one more factor gives a factor of 3125, which is more than 1000.

(a) At most 4 factors. (b) Within a factor of 625.

4. Some books give "order of magnitude" a second meaning: round to the nearest power of ten on the power-of-ten scale, where the cut-off between 10^n and 10^{n+1} is $\sqrt{10} \times 10^n \approx 3.16 \times 10^n$.

(a) Give the order of magnitude of each number in the lesson's sense and in this second sense: 2.5×10^4, 400, 0.07, 3.2×10^8.

(b) Ordinary rounding would send 3.2×10^8 down to 10^8. Explain why the second sense sends it up instead.

ANSWER

(a) Lesson's sense: 10^4, 10^2, 10^{-2}, 10^8. Second sense: 10^4, 10^3, 10^{-1}, 10^9. (b) 3.2 is more than $\sqrt{10} \approx 3.16$, so on the power-of-ten scale 3.2×10^8 is closer to 10^9 than to 10^8.

SOLUTION

(a) In the lesson's sense, write $a \times 10^n$ with $1 \le a < 10$ and take 10^n: 2.5×10^4 gives 10^4; $400 = 4 \times 10^2$ gives 10^2; $0.07 = 7 \times 10^{-2}$ gives 10^{-2}; 3.2×10^8 gives 10^8.

In the second sense, compare a with $\sqrt{10} \approx 3.16$: below it stays at 10^n, above it goes up to 10^{n+1}. So $2.5 < 3.16$ gives 10^4; $4 > 3.16$ gives 10^3; $7 > 3.16$ gives $10^{-2+1} = 10^{-1}$; $3.2 > 3.16$ gives 10^9.

(b) On the power-of-ten scale, 3.2×10^8 is 3.2 times 10^8 but only $\dfrac{10}{3.2} \approx 3.1$ times below 10^9, so it is nearer 10^9. The point halfway between 10^8 and 10^9 is $\sqrt{10^8 \times 10^9} = \sqrt{10} \times 10^8$, and $3.2^2 = 10.24 > 10$ shows 3.2 is just past it. Ordinary rounding uses the ordinary halfway point, 5.5×10^8, instead.

5. A high school has 1200 students. Assume each student sends at least 3 and at most 192 text messages on a school day.

(a) Find the middle of this range on the power-of-ten scale.

(b) Use it to estimate how many texts the whole school sends on a school day, and give low and high bounds for that total.

ANSWER

(a) 24 texts per student (b) About 28,800 texts, with bounds 3,600 and 230,400 texts.

SOLUTION

(a) The middle of the range from 3 to 192 is $\sqrt{3 \times 192} = 24$ texts per student per day. It is 8 times the low end and $\dfrac{1}{8}$ of the high end.

(b) $1200 \text{ students} \times \dfrac{24 \text{ texts}}{1 \text{ student}}$ gives about 28,800 texts on a school day.

All the factors are positive, so the bounds come from the two ends of the range: 1200 × 3 is 3,600 and 1200 × 192 is 230,400 texts.

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6. An above-ground pool holds between 4800 and 12960 liters of water. A garden hose delivers between 12 and 40 liters a minute.

(a) What are the fewest and the most hours it could take to fill the pool with the hose?

(b) Find the middle of that range of times on the power-of-ten scale.

ANSWER

(a) Fewest 2 hours, most 18 hours (b) 6 hours

SOLUTION

(a) Time is volume divided by rate. The rate divides, so the fewest hours use the smallest pool and the fastest hose, and the most hours use the largest pool and the slowest hose:

$$\text{fewest: } \frac{4800\ \text{L}}{40\ \text{L/min}} \times \frac{1\ \text{h}}{60\ \text{min}} = 2\ \text{h}, \qquad \text{most: } \frac{12960\ \text{L}}{12\ \text{L/min}} \times \frac{1\ \text{h}}{60\ \text{min}} = 18\ \text{h}$$

(b) $\sqrt{2 \times 18} = 6$ hours.

7. Kevin estimates how many pet dogs live in a city. His assumptions, each with a low and a high value:

- the city has between 2.0 million and 2.4 million people;
- a household has between 2 and 3 people;
- between one household in 10 and one household in 2 has a dog (count one dog each).

(a) Write the chain for the number of dogs, with its units.

(b) Find low and high bounds for the number of dogs.

(c) How many times the low bound is the high bound?

(d) Kevin has time to research one assumption. Which should it be? Explain using how many times its high value is its low value.

ANSWER

(a) $\text{dogs} \approx \text{people} \times \dfrac{1\ \text{household}}{\text{people per household}} \times \dfrac{\text{dogs}}{\text{household}}$ (b) low about 6.7×10^4, high 6×10^5 (c) 9 times (d) the share of households with a dog

SOLUTION

(a)

$$\text{dogs} \approx \text{people} \times \frac{1\ \text{household}}{h\ \text{people}} \times \frac{1\ \text{dog}}{d\ \text{households}}$$

where h is the people per household and one household in d has a dog. People cancel, households cancel, leaving dogs.

(b) Two factors divide, so the low bound uses the fewest people, the largest households and the rarest dogs:

$$\text{low: } 2.0 \times 10^6 \times \frac{1}{3} \times \frac{1}{10} \approx 66\,667 \approx 6.7 \times 10^4$$

$$\text{high: } 2.4 \times 10^6 \times \frac{1}{2} \times \frac{1}{2} = 600\,000 = 6 \times 10^5$$

(c) $600\,000 \div 66\,667 \approx 9$; exactly, $\dfrac{2.4}{2.0} \times \dfrac{3}{2} \times \dfrac{10}{2} = 1.2 \times 1.5 \times 5 = 9$.

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(d) The three ranges have high-to-low ratios 1.2 (people), 1.5 (household size) and 5 (share with a dog). The answer's range is their product, and almost all of it comes from the dog share. Narrowing the population would barely help; pinning down the share of households with a dog would shrink the range most.

8. Hannah estimates how many pizzas are ordered in one week in a town of 30,000 people, assuming that about one person in 8 orders one pizza a week. The town's pizzerias later report 15,000 pizzas that week.

   (a) Find Hannah's estimate.

   (b) How many times the estimate is the reported figure?

   (c) The report suggests a better value for the assumption "one person in 8". What is it?

   ANSWER

   (a) 3,750 pizzas (b) 4 times (c) one person in 2

   SOLUTION

   (a) 30,000 people $\times \dfrac{1 \text{ pizza}}{8 \text{ people}}$ gives 3,750 pizzas a week.

   (b) 15,000 divided by 3,750 is 4, so the estimate is low by a factor of 4.

   (c) The number of people per pizza that matches the report is 30,000 divided by 15,000, which is 2: about one person in 2 orders a pizza a week. That assumption was the weak link.

9. Estimate how many sheets of printer paper, stacked flat, would reach the Moon, about 384 000 km away. State the thickness you assume for one sheet, with a low and a high value for it, and give your estimate in scientific notation with low and high bounds.

   ANSWER

   One reasonable chain: a sheet is about 0.1 mm thick (surely between 0.05 and 0.2 mm), giving about 4×10^{12} sheets, with bounds about 2×10^{12} and 8×10^{12}. Other stated assumptions give other numbers of the same order.

   SOLUTION

   Assumption: a pack of 500 sheets is about 5 cm thick, so one sheet is about $\dfrac{50 \text{ mm}}{500} = 0.1 \text{ mm}$. It is surely between 0.05 mm and 0.2 mm.

   Put the distance in millimeters:

   $$3.84 \times 10^5 \text{ km} \times \frac{10^3 \text{ m}}{1 \text{ km}} \times \frac{10^3 \text{ mm}}{1 \text{ m}} = 3.84 \times 10^{11} \text{ mm}$$

   Then divide by the thickness of one sheet:

   $$\frac{3.84 \times 10^{11} \text{ mm}}{0.1 \text{ mm per sheet}} = 3.84 \times 10^{12} \approx 4 \times 10^{12} \text{ sheets}$$

   The thickness divides, so the low bound uses the thick guess and the high bound the thin one: $\dfrac{3.84 \times 10^{11}}{0.2} = 1.92 \times 10^{12} \approx 2 \times 10^{12}$ and $\dfrac{3.84 \times 10^{11}}{0.05} = 7.68 \times 10^{12} \approx 8 \times 10^{12}$ sheets. A few trillion sheets.

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