Combining transformations

A rule like $y = -2(x-5)^3 + 7$ shifts, stretches and flips a parent graph all at once. How do you read every move off the rule, in the right order, and say exactly where each point of the parent lands?

WORDS TO KNOW

Parent function f: the plain graph you start from, such as $y = x^2$ or $y = \sqrt{x}$.

Outside numbers a and k: they act on the output, the height.

Image: the point a parent point lands on. "$\mapsto$" means "goes to".

General form: $g(x) = a\,f\big(b(x-h)\big) + k$, with $a \neq 0$ and $b \neq 0$.

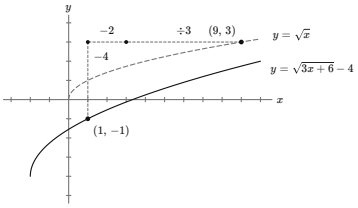
Inside numbers b and h: they act on the input, the left-right position.

KEY IDEAS

1. Each number does one job

Number	Where it sits	What it does to the graph
a	outside, multiplies	stretch (or squeeze) up and down by $\lvert a\rvert$; flip across the x-axis if $a < 0$
k	outside, added	shift up k (down if $k < 0$)
b	inside, multiplies	squeeze (or stretch) left and right by $\dfrac{1}{\lvert b\rvert}$; flip across the y-axis if $b < 0$
h	inside, subtracted	shift right h (left if $h < 0$)

2. Where one point goes, and the order of the moves



A point (p, q) of f goes to

$$\left(\frac{p}{b} + h,\;\; aq + k\right)$$

Height: multiply by a, **then** add k. Position: divide by b, **then** add h. In each direction the stretch or flip comes first and the shift second.

Horizontal moves never change a height, and vertical moves never change a position. So the two directions do not interfere: only the order within each direction matters.

3. Get the inside into the form $b(x-h)$ first

You can read h only when b multiplies everything in the parentheses. Factor first: $2x + 6 = 2(x+3)$, so $h = -3$, a shift of 3 left, not 6. A minus sign in front of x is $b = -1$: $7 - x = -(x-7)$, so $h = 7$.

To run a point (x, y) of g back to f, undo the steps in reverse order: the parent point is $\left(b(x-h),\; \dfrac{y-k}{a}\right)$.

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WATCH OUT

- **Factor before you read the shift.** In $\sqrt{3x-15}$ the shift is not 15. Since $3x-15=3(x-5)$, it is 5 right.
- **The stretch comes before the shift.** A shift done before a stretch gets stretched too. Doubling then adding 5 sends q to $2q+5$; adding 5 then doubling gives $2q+10$.
- **The inside is undone backwards.** The rule subtracts h and then multiplies by b, so to find the new x you divide by b and then add h.
- **Check one image in the rule.** Put its x into the rule and make sure you get its y.

WORKED EXAMPLES

Example 1. Describe, in order, how $y=-3(x-4)^2+2$ is built from $y=x^2$. Carry the parent's points (−1, 1), (0, 0), (1, 1) and (2, 4) across.

1. Match with $a\,f\big(b(x-h)\big)+k$: $a=-3$, $b=1$, $h=4$, $k=2$.
2. The moves: shift right 4. Stretch vertically by 3 and flip across the x-axis, then shift up 2.
3. Each point goes to $(p+4,\ -3q+2)$. So $(-1,1)\mapsto(3,-1)$, $(0,0)\mapsto(4,2)$, $(1,1)\mapsto(5,-1)$ and $(2,4)\mapsto(6,-10)$.

Answer: right 4; stretch by 3 and flip; up 2. The images are (3, −1), (4, 2), (5, −1), (6, −10). The vertex (4, 2) is now the highest point. **Check:** $-3(6-4)^2+2=-12+2=-10$.

Example 2. Describe how $y=\sqrt{5x+10}-3$ is built from $y=\sqrt{x}$, and carry (0, 0), (1, 1) and (25, 5) across.

1. Factor the inside: $5x+10=5(x+2)$. So $a=1$, $b=5$, $h=-2$, $k=-3$.
2. The moves: compress horizontally by a factor of $\frac{1}{5}$, then shift left 2. Shift down 3.
3. Each point goes to $\left(\frac{p}{5}-2,\ q-3\right)$. So $(0,0)\mapsto(-2,-3)$, $(1,1)\mapsto(-1.8,-2)$ and $(25,5)\mapsto(3,2)$.

Answer: compress by $\frac{1}{5}$, left 2, down 3; images $(-2,-3)$, $(-1.8,-2)$, $(3,2)$. **Check:** at $x=3$, $\sqrt{25}-3=2$.

Example 3. Start from $y=\sqrt{x}$. Lena shifts it up 4 and then compresses it vertically by a factor of $\frac{1}{2}$. Omar does the same two moves in the other order. Write both rules and follow the point (9, 3).

1. A vertical compression multiplies the **whole** current output. Lena's output after her shift is $\sqrt{x}+4$, so her rule is $y=\frac{1}{2}(\sqrt{x}+4)=\frac{1}{2}\sqrt{x}+2$.
2. Omar compresses first and then adds 4: $y=\frac{1}{2}\sqrt{x}+4$.
3. Lena: (9, 3) goes up to (9, 7), then to half the height, (9, 3.5). Omar: (9, 3) goes to (9, 1.5), then up to (9, 5.5).

Answer: Lena $y=\frac{1}{2}\sqrt{x}+2$, Omar $y=\frac{1}{2}\sqrt{x}+4$. The graphs differ by 2 at every input. The k of the general form is the shift done last. **Check:** $\frac{1}{2}\sqrt{9}+2=3.5$ and $\frac{1}{2}\sqrt{9}+4=5.5$.

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