Graphing with transformations

A rule like $y = 2(x+3)^2 + 1$ is a moved copy of a parent graph. How do you draw it by moving a few of the parent's points, and how do you find its domain and range?

WORDS TO KNOW

Parent graph: the plain graph a rule is built from, such as $y = x^2$ or $y = \dfrac{1}{x}$.

Asymptote: a line the graph gets closer and closer to without ever reaching it.

Interval notation: $[3, \infty)$ means $x \geq 3$; $(-\infty, 3]$ means $x \leq 3$. A square bracket includes the end, a parenthesis leaves it out.

Anchor: the parent's key point: the vertex of x^2, the center of x^3, the endpoint of $\sqrt{x}$, or the crossing of the asymptotes of $\dfrac{1}{x}$.

Domain and **range:** all the inputs a rule accepts, and all the outputs it gives.

KEY IDEAS

1. Carry a few points, then draw the parent's shape

Write the rule as $y = a\,f\big(b(x-h)\big) + k$, factoring the inside first. Every point of the parent moves by one rule, so the whole graph moves as one piece.

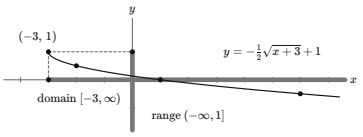
Each parent point (p, q) lands at $\left(\dfrac{p}{b} + h,\ aq + k\right)$. The anchor (0, 0) lands at (h, k).

For $\dfrac{1}{x}$, the asymptotes move to $x = h$ and $y = k$. Only the shifts move them.

A negative a turns the graph upside down. A negative b turns it left for right.

Parent	Anchor	Easy points
$y = x^2$	vertex (0, 0)	$(\pm 1, 1)$, $(\pm 2, 4)$
$y = x^3$	center (0, 0)	$(\pm 1, \pm 1)$, $(\pm 2, \pm 8)$
$y = \sqrt{x}$	endpoint (0, 0)	(1, 1), (4, 2), (9, 3)
$y = \dfrac{1}{x}$	asymptotes $x = 0$, $y = 0$	(1, 1), (−1, −1), $\left(2, \dfrac{1}{2}\right)$

2. The domain and range move too



The domain lists inputs, so only b and h can move it. The range lists outputs, so only a and k can move it. Carry the ends of each interval.

- $y = x^2$: range $[k, \infty)$ if $a > 0$, $(-\infty, k]$ if $a < 0$.
- $y = \sqrt{x}$: domain $[h, \infty)$ if $b > 0$, $(-\infty, h]$ if $b < 0$; range as for x^2.
- $y = \dfrac{1}{x}$: every number except h, and every number except k.
- $y = x^3$: all numbers, for both.

A negative number swaps the two ends of an interval.

Free sample from math.wiki for classroom use. Do not post. math.wiki

WATCH OUT

- **Factor the inside first.** Only $b(x-h)$ shows h. In $(3x-9)^2$ the vertex is at $x=3$, since $3x-9=3(x-3)$.
- **A stretch does not move an asymptote.** The asymptotes of $\dfrac{1}{x}$ move only with h and k.
- **With $a<0$ the endpoint is the top.** For $y=-\sqrt{x}+2$ the range is $(-\infty,2]$, not $[2,\infty)$.
- **Test one point in the rule** before you join the points.

WORKED EXAMPLES

Example 1. Graph $y=\dfrac{1}{2}(x+2)^2-5$ by moving $y=x^2$.

1. Read the numbers: $a=\dfrac{1}{2}$, $b=1$, $h=-2$, $k=-5$. The vertex goes to $(h,k)=(-2,-5)$.
2. Each point goes to $\left(p-2,\ \dfrac{q}{2}-5\right)$. So $(\pm2,4)$ go to (0, −3) and (−4, −3), and $(\pm4,16)$ go to (2, 3) and (−6, 3).
3. Since $a>0$ it still opens up. Since $a=\dfrac{1}{2}$ it is wider than the parent.
4. Join the five points with a parabola whose axis is the line $x=-2$.

Answer: an upward parabola with vertex (−2, −5) through $(0,-3)$, $(-4,-3)$, $(2,3)$ and $(-6,3)$. **Check:** $\frac{1}{2}(2+2)^2-5=8-5=3$.

Example 2. Graph $y=\dfrac{3}{x+7}-2$ by moving $y=\dfrac{1}{x}$, and give its domain and range.

1. Since $\dfrac{3}{x+7}=3\cdot\dfrac{1}{x+7}$: $a=3$, $b=1$, $h=-7$, $k=-2$.
2. Draw the asymptotes first, dashed: $x=-7$ and $y=-2$. They cross at (−7, −2).
3. Each point goes to $(p-7,\ 3q-2)$: $(1,1)\mapsto(-6,1)$, $(-1,-1)\mapsto(-8,-5)$, $(3,\dfrac{1}{3})\mapsto(-4,-1)$.
4. With $a>0$ there is no flip: one branch sits up and to the right of the crossing, the other down and to the left.

Answer: asymptotes $x=-7$ and $y=-2$. Domain $(-\infty,-7)\cup(-7,\infty)$, range $(-\infty,-2)\cup(-2,\infty)$. **Check:** $\dfrac{3}{-4+7}-2=1-2=-1$.

Example 3. Find the domain and range of $y=-\sqrt{8-x}+1$, and sketch it.

1. Factor the inside: $8-x=-(x-8)$. So $a=-1$, $b=-1$, $h=8$, $k=1$. The endpoint is (8, 1).
2. Domain: $b<0$ swaps the ends of $[0,\infty)$, giving $(-\infty,8]$. Directly, $8-x\geq0$ means $x\leq8$.
3. Range: $a<0$ swaps the ends of $[0,\infty)$, and then 1 is added, giving $(-\infty,1]$.
4. Points go to $(8-p,\ 1-q)$: $(1,1)\mapsto(7,0)$, $(4,2)\mapsto(4,-1)$, $(9,3)\mapsto(-1,-2)$. The graph runs left and down from the endpoint.

Answer: domain $(-\infty,8]$, range $(-\infty,1]$. **Check:** $-\sqrt{8-(-1)}+1=-3+1=-2$.

Free sample from math.wiki for classroom use. Do not post. math.wiki