Linear functions in context

A situation changes by the same amount at every step. How do you turn its description into a function, and what do the two numbers in that function say about the situation?

WORDS TO KNOW

Linear function: $f(x) = mx + b$. The output changes by the same amount for each step of the input.

Rate of change (m): the change in output for each 1 unit of input. Its units are output units **per** input unit.

Domain: the inputs that make sense in the situation.

Starting value (b): the output when the input is 0, $b = f(0)$. It has the output's units.

Input and output: the output is the quantity that depends on the other one.

KEY IDEAS

1. Find the two numbers

From words: a fixed amount is b; an amount "per unit" is m. Two charges per unit on the same input add into one rate.

From two pairs (x_1, y_1) and (x_2, y_2): $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, then walk back to input 0: $b = y_1 - mx_1$.

From a table: check that every pair of neighboring rows gives the same rate. If the input steps by 3, divide the change by 3.

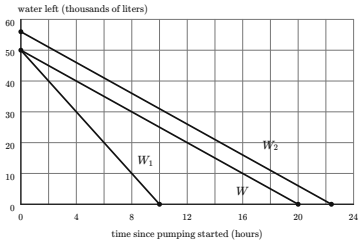
From a graph: b is where the line meets the vertical axis (if the horizontal axis starts at 0). Get m from two grid corners.

Always test the rule on a third row, point or fact.

2. Changing a number

Raise b by k: every output rises by k. The line moves straight up, parallel to the old one.

Change m by k: the output at x changes by kx. The line pivots about $(0, b)$.



3. Comparing two functions

The starting values say who is ahead at input 0. The rates say who gains faster. If the one behind gains faster, the gap closes after $\dfrac{\text{starting gap}}{\text{difference in rates}}$ units. Equal rates mean the gap never closes.

WATCH OUT

- **Forgetting the units.** "24" means nothing; "24 dollars per seat" does.
- **Forgetting to divide by the step.** A change of 7500 over 3 hours is 2500 per hour, not 7500.
- **Doubling the rate does not double the output.** It doubles only mx, because b stays the same.
- **The starting value may not be a real case.** Nobody hires a car for 0 days; $C(0)$ is just the fixed part of every price.

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WORKED EXAMPLES

Example 1. A car hire costs 38 dollars a day plus 9 dollars a day for insurance, plus a one-time cleaning fee of 20 dollars. Write the cost $C(d)$ for d days, say what the numbers mean, and find $C(4)$.

1. Input: days d. Output: cost in dollars.
2. Each day brings two charges: 38 + 9 = 47 dollars per day. That is m.
3. The cleaning fee is paid once, however long the hire. That is $b = 20$ dollars.
4. $C(d) = 47d + 20$, so $C(4) = 188 + 20 = 208$.

Answer: $C(d) = 47d + 20$; a 4 day hire costs 208 dollars. Each day adds 47 dollars; 20 dollars is fixed.

Example 2. A theater show that sold 120 seats made a profit of 1080 dollars; one that sold 200 seats made 3000 dollars. The profit is linear in seats sold. Write $P(s)$ and find the break-even number of seats.

1. Rate, in dollars per seat:
$$m = \frac{3000 - 1080}{200 - 120} = \frac{1920}{80} = 24$$
2. Walk back from (120, 1080): $b = 1080 - 24(120) = 1080 - 2880 = -1800$.
3. $P(s) = 24s - 1800$. Each seat adds 24 dollars; a show with no seats sold loses 1800 dollars, the fixed costs.
4. Break even: $24s - 1800 = 0$, so $24s = 1800$ and $s = 75$.

Answer: $P(s) = 24s - 1800$; 75 seats. **Check:** 24(200) − 1800 = 3000.

Example 3. A water tower is pumped empty. The water left is 45000 liters at 2 hours, 37500 at 5 hours, 30000 at 8 hours and 22500 at 11 hours. Write $W(t)$ and find when the tower is empty.

1. The input steps by 3 hours, and the water drops 7500 liters each step. Rate: −7500 ÷ 3 = −2500 liters per hour.
2. There is no row for $t = 0$. Walk back 2 hours from the first row: 45000 + 2(2500) = 50000.
3. $W(t) = 50000 - 2500t$. The tower held 50000 liters at the start.
4. Empty: $2500t = 50000$, so $t = 20$.

Answer: $W(t) = 50000 - 2500t$; empty after 20 hours. **Check:** $W(11) = 50000 - 27500 = 22500$.

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