Reflecting a graph across the axes

One minus sign turns a graph over. Where you put it, outside on the output or inside on the input, decides which way it flips.

WORDS TO KNOW

Reflection: a flip of the graph across a line, the mirror. Each point lands the same distance away on the other side.

$-f(x)$**:** take the output of f, then change its sign.

$f(-x)$**:** change the sign of the input first, then use f.

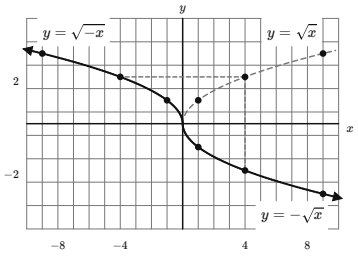
Zero of f: an input where $f(x) = 0$, where the graph meets the x-axis.

Domain and **range:** all the inputs a function accepts, and all the outputs it gives.

KEY IDEAS

1. A minus sign outside flips the graph across the x-axis

In $g(x) = -f(x)$ the input is untouched and every output changes sign. A point above the x-axis lands the same distance below it, at the same x.



$y = -f(x)$: $(a, b) \mapsto (a, -b)$, a reflection across the x-axis.

The zeros stay put. The domain stays the same. The range is replaced by its opposites.

2. A minus sign inside flips the graph across the y-axis

In $h(x) = f(-x)$ the sign changes before f works. To hand f the number a, the input must be $-a$. So the height b that f had at a now appears at $-a$.

$y = f(-x)$: $(a, b) \mapsto (-a, b)$, a reflection across the y-axis.

The y-intercept stays put. The range stays the same. The domain is replaced by its opposites.

3. Both flips at once, and naming a flip

$y = -f(-x)$: $(a, b) \mapsto (-a, -b)$. This is a half turn about the origin. The order of the two flips does not matter.

To name a flip from a graph, compare matching points. Upside down points to the outside sign; facing the other way points to the inside sign. Some flips change nothing: $(-x)^2 = x^2$, so the parabola $y = x^2$ is its own mirror image across the y-axis.

WATCH OUT

- $-x^2$ **means** $-(x^2)$**.** The power is done first, so $y = -x^2$ opens down. $(-x)^2$ is just x^2.
- **The sign must reach the whole output.** Flipping $y = \sqrt{x} + 5$ across the x-axis gives $-(\sqrt{x} + 5) = -\sqrt{x} - 5$, not $-\sqrt{x} + 5$.
- **In** $f(-x)$**, replace every** x**.** Every x in the rule changes sign, including any inside parentheses or a root. Do not just flip the signs of some terms.
- **Two different flips can give the same graph.** For $y = x^3$ and $y = \dfrac{1}{x}$, both flips give one graph, so both names are right.

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WORKED EXAMPLES

Example 1. Let $f(x) = \sqrt{x+2}$. Write $-f(x)$ and $f(-x)$, give the domain and range of each, and say where the point (2, 2) of f goes.

1. Outside sign: $-f(x) = -\sqrt{x+2}$. The inputs are the same, so the domain is still $x \geq -2$. The outputs $y \geq 0$ become $y \leq 0$. The point (2, 2) goes to (2, −2).
2. Inside sign: replace x by $-x$. $f(-x) = \sqrt{-x+2} = \sqrt{2-x}$.
3. The root needs $2 - x \geq 0$, so the domain is $x \leq 2$. The outputs are the same, so the range is still $y \geq 0$. The point (2, 2) goes to (−2, 2).

Answer: $-f(x) = -\sqrt{x+2}$, domain $x \geq -2$, range $y \leq 0$, point $(2, -2)$. $f(-x) = \sqrt{2-x}$, domain $x \leq 2$, range $y \geq 0$, point $(-2, 2)$. **Check:** $-\sqrt{2+2} = -2$ and $\sqrt{2-(-2)} = 2$.

Example 2. Graph $y = -(x+5)^2$ from $y = x^2$. Give the vertex, the direction it opens and the range.

1. Work from the inside out. $(x+5)^2$ is $y = x^2$ shifted 5 left, with vertex (−5, 0).
2. The minus sign in front acts on the whole output, so it flips that parabola across the x-axis.
3. The vertex is on the mirror, so it stays at (−5, 0). Every other point drops below the axis, so the parabola opens down.
4. A square is never negative, so $-(x+5)^2 \leq 0$, with 0 only at $x = -5$.

Answer: vertex (−5, 0), the highest point; opens down; range $y \leq 0$. **Check:** at $x = -3$, $-(2)^2 = -4$, and at $x = -7$, $-(-2)^2 = -4$: two points at the same height, one on each side of the vertex.

Example 3. Reflect $y = x^3$ across the x-axis, then across the y-axis instead. Compare the two graphs.

1. Across the x-axis: $-f(x) = -x^3$. The parent's (2, 8) goes to (2, −8).
2. Across the y-axis: $f(-x) = (-x)^3 = (-x)(-x)(-x) = -x^3$, since three negative factors give a negative product. The point (2, 8) goes to (−2, 8).
3. The two rules are the same function, so the two flips give the same graph. It falls from upper left to lower right.

Answer: both give $y = -x^3$. **Check:** (2, −8) and (−2, 8) both fit $y = -x^3$: $-(2^3) = -8$ and $-((-2)^3) = 8$.

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