The squaring and cubing parent functions

Squaring a number forgets its sign, and cubing keeps it. What does that one difference do to the graphs of $y = x^2$ and $y = x^3$?

WORDS TO KNOW

Squaring function: $f(x) = x^2$, which multiplies the input by itself.

Cubing function: $g(x) = x^3$, which multiplies the input by itself twice.

Parabola: the U-shaped graph of $y = x^2$.

Vertex: the turning point of the U. For $y = x^2$ it is (0, 0), the lowest point.

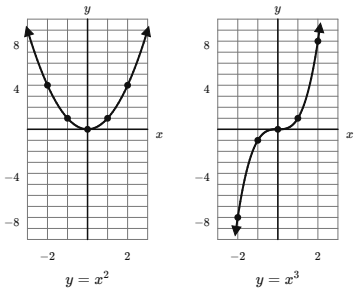
Domain: all the inputs a function accepts. **Range:** all the outputs it gives.

Cube root: $\sqrt[3]{c}$ is the one number whose cube is c.

KEY IDEAS

1. Squaring forgets the sign, so the U stays on or above the axis

A number and its opposite have the same square: $(-a)^2 = a^2$. So every point (a, a^2) has a partner $(-a, a^2)$ at the same height. So the U is its own mirror image, with the y-axis as the mirror. No square is negative.



$y = x^2$ passes through (−2, 4), (−1, 1), (0, 0), (1, 1), (2, 4).

Domain: all real numbers. Range: $y \geq 0$. It falls for $x \leq 0$ and rises for $x \geq 0$, so the vertex (0, 0) is the lowest point.

Each height $c > 0$ is reached twice, at $x = \sqrt{c}$ and $x = -\sqrt{c}$. The height 0 is reached once, and no negative height is reached.

2. Cubing keeps the sign, so the curve runs from lower left to upper right

Three negative factors give a negative product: $(-a)^3 = -a^3$. So negative inputs give negative outputs.

$y = x^3$ passes through (−2, −8), (−1, −1), (0, 0), (1, 1), (2, 8).

Domain and range: all real numbers. It rises everywhere, so it has no highest or lowest point.

Each height c is reached exactly once, at $x = \sqrt[3]{c}$. The curve is flat for an instant at the origin, but it does not turn there.

3. Near 0 flat, far from 0 steep

Multiplying a positive number by a number between 0 and 1 makes it smaller, so near the origin both graphs hug the x-axis. Past 1, each extra factor makes the output larger.

For $0 < x < 1$: $x^3 < x^2 < x$. For $x > 1$: $x < x^2 < x^3$.

The graphs of $y = x^2$ and $y = x^3$ meet only at (0, 0) and (1, 1).

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WATCH OUT

- **The ends do not always give the range.** On a stretch of inputs that includes 0, the lowest output of $y = x^2$ is 0, at the vertex. The highest comes from the input farthest from 0.
- **$x^2 = c$ has two answers when $c > 0$.** The sign $\sqrt{c}$ names only the positive one, so also give $-\sqrt{c}$.
- **Keep the sign when you cube.** A negative number cubed is negative, so $(-a)^3$ is never the same as a^3 unless $a = 0$.
- **Flat is not turning.** $y = x^3$ is level at the origin but rises on both sides of it. A turning point rises on one side and falls on the other, like the vertex of the U.

WORKED EXAMPLES

Example 1. Find x^2 and x^3 at $x = -1.1$ and at $x = 0.8$. Which points lie below the x-axis?

1. Square first: $(-1.1)^2 = 1.21$, positive because two negative factors give a positive product.
2. Multiply by the input once more: $(-1.1)^3 = 1.21 \cdot (-1.1) = -1.331$.
3. At 0.8: $0.8^2 = 0.64$ and $0.8^3 = 0.64 \cdot 0.8 = 0.512$.

Answer: the points are $(-1.1, 1.21)$ and $(0.8, 0.64)$ on the parabola, $(-1.1, -1.331)$ and $(0.8, 0.512)$ on the cubic. Only $(-1.1, -1.331)$ is below the axis, because cubing kept the sign. **Check:** at 0.8, which is between 0 and 1, the order is 0.512 < 0.64 < 0.8, so $x^3 < x^2 < x$.

Example 2. Find every x with $x^2 = 1.69$, with $x^2 = -9$, and with $x^3 = 2.197$.

1. Each question asks where a horizontal line meets a graph.
2. The line $y = 1.69$ meets the U twice. Since $1.3^2 = 1.69$, the inputs are 1.3 and −1.3.
3. The line $y = -9$ lies wholly below the U, because no square is negative. There is no solution.
4. The cubic rises everywhere, so it meets each height once. Since $1.3^3 = 1.69 \cdot 1.3 = 2.197$, the only input is 1.3. The input −1.3 gives −2.197 instead.

Answer: $x = 1.3$ or $x = -1.3$; no solution; $x = 1.3$ only.

Example 3. Find the range of $y = x^2$ and of $y = x^3$ when the inputs are limited to $-1.6 \le x \le 1.1$.

1. Parabola: the input 0 is inside the stretch, so the lowest output is the vertex height 0.
2. The input farthest from 0 is −1.6, not 1.1. Its output is $(-1.6)^2 = 2.56$, more than $1.1^2 = 1.21$. So the range is $0 \le y \le 2.56$.
3. Cubic: it always rises, so the least output is at the left end and the greatest at the right end: $(-1.6)^3 = -4.096$ and $1.1^3 = 1.331$.

Answer: $0 \le y \le 2.56$ for the parabola; $-4.096 \le y \le 1.331$ for the cubic.

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