Testing for symmetry with algebra: even and odd functions

A picture can only suggest that a graph is symmetric. How do you use the rule to prove it, for every input at once?

WORDS TO KNOW

Even function: $f(-x) = f(x)$ for every x in the domain. Its graph is symmetric about the y-axis.

Neither: a function that is not even and not odd. This is the usual case.

$f(-x)$**:** the rule with $-x$ put in for every x.

Odd function: $f(-x) = -f(x)$ for every x in the domain. Its graph is symmetric about the origin.

Symmetric domain: whenever x is in the domain, so is $-x$.

KEY IDEAS

1. The test is a calculation

Put $-x$ in for every x and simplify $f(-x)$.

- If $f(-x) = f(x)$: **even**.
- If $f(-x) = -f(x)$, the opposite of every term: **odd**.
- Otherwise: **neither**.

Check the domain first. If some input has no opposite in the domain, the function is neither.

To show a function is **not** even, or **not** odd, one input is enough: find a with $f(-a) \neq f(a)$, or with $f(-a) \neq -f(a)$. To show it **is** even or odd, you need the algebra for a general x. A few inputs that work prove nothing.

2. Where the names come from

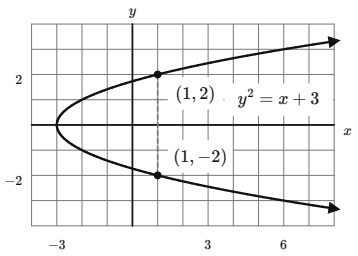
$(-x)^n = x^n$ when n is even, and $(-x)^n = -x^n$ when n is odd. So:

- A polynomial with only even powers (a constant counts as even) is even.
- A polynomial with only odd powers is odd.
- A polynomial with both kinds is neither.
- Products: even times even is even, odd times odd is even, even times odd is odd.

Beyond polynomials: $|x|$ is even, since $|-x| = |x|$, and $\dfrac{1}{x}$ is odd.

3. Equations that are not functions

For an equation in x and y, ask whether the partner of each point still makes it true.



Replace	Partner of (a, b)	Same equation means symmetry about
y by $-y$	$(a, -b)$	the x-axis
x by $-x$	$(-a, b)$	the y-axis
both	$(-a, -b)$	the origin

When the equation changes, prove it with one point that fits the old equation and fails the new one.

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WATCH OUT

- **Change the sign of every x.** In $f(-x)$, each x in the rule becomes $-x$, including those inside powers, roots and bars.
- **Two inputs that match prove nothing.** The rule must hold for every x, so $f(-3) = f(3)$ alone does not show that f is even.
- **Even and odd are not opposites.** Most functions are neither. Only $f(x) = 0$ is both.
- **A shifted graph usually loses its symmetry.** $y = (x + 5)^2$ is neither: its mirror is the line $x = -5$, not the y-axis.

WORKED EXAMPLES

Example 1. Decide whether each function is even, odd or neither: (a) $f(x) = 3x^4 - x^2 + 2$, (b) $g(x) = -x^5 + 6x^3$, (c) $h(x) = x^3 - 2x^2$.

1. (a) All the powers are even (the 2 is $2x^0$), so each term keeps its sign: $f(-x) = 3x^4 - x^2 + 2 = f(x)$. So f is even.
2. (b) All the powers are odd, so each term changes sign: $g(-x) = x^5 - 6x^3$. That is $-(-x^5 + 6x^3) = -g(x)$. So g is odd.
3. (c) $h(-x) = -x^3 - 2x^2$, which is neither $h(x)$ nor $-h(x)$. Prove it with one input: $h(2) = 8 - 8 = 0$ and $h(-2) = -8 - 8 = -16$.
4. Since $-16 \neq 0$, h is not even. Since $-16 \neq -0$, it is not odd.

Answer: (a) even, (b) odd, (c) neither. **Check (b):** $g(2) = -32 + 48 = 16$ and $g(-2) = 32 - 48 = -16$, opposites.

Example 2. Decide whether each function is even, odd or neither: (a) $r(x) = \dfrac{x^2 - 1}{x^4 + 2}$, (b) $s(x) = \dfrac{1}{x + 4}$.

1. (a) The denominator is never 0, so the domain is every real number, which is symmetric.
2. Every x appears to an even power: $r(-x) = \dfrac{(-x)^2 - 1}{(-x)^4 + 2} = \dfrac{x^2 - 1}{x^4 + 2} = r(x)$. So r is even.
3. (b) Look at the domain first. s has no value at $x = -4$, but $s(4) = \dfrac{1}{8}$ exists. The input 4 has no partner in the domain.

Answer: (a) even; (b) neither, because its domain is not symmetric. No algebra is needed for (b).

Example 3. Test the graph of $x = y^2 - 4$ for symmetry about the x-axis, the y-axis and the origin.

1. x-axis: replace y by $-y$. Since $(-y)^2 = y^2$, the result is $x = y^2 - 4$ again. So it is symmetric about the x-axis. For example (0, 2) and (0, −2) are both on it.
2. y-axis: replace x by $-x$, giving $-x = y^2 - 4$, a different equation. The point (−4, 0) is on the graph, since $0 - 4 = -4$. Its partner (4, 0) is not: $0 - 4 \neq 4$.
3. Origin: replace both, giving $-x = (-y)^2 - 4$, which is again $-x = y^2 - 4$. The partner of (−4, 0) is (4, 0), which fails, as in step 2.

Answer: symmetric about the x-axis only. It is a parabola opening to the right from (−4, 0), so it is not the graph of a function.

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