Transforming the absolute value graph

The graph of $y = a|x - h| + k$ is the V of $y = |x|$ moved, stretched or flipped. How do you read its corner, its arms and its range, write its rule from a picture, and solve equations off it?

WORDS TO KNOW

Absolute value $|x|$: the distance of x from 0. So $|-3| = 3$ and $|3| = 3$.

Arm: one of the two straight halves of the V. Each is a ray.

Range: all the outputs the graph reaches.

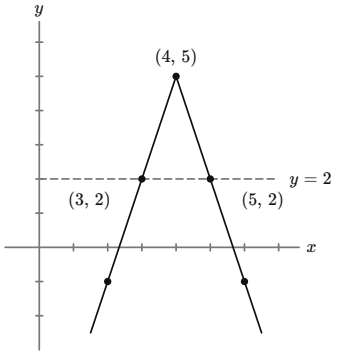
Vertex: the corner of the V, its lowest or highest point.

Slope: how much the height changes for each step of 1 to the right.

KEY IDEAS

1. The vertex is (h, k) and the arms have slopes a and $-a$

The bars $|x - h|$ are 0 at $x = h$, so the corner sits at (h, k). To the right of it the rule is the line $y = a(x - h) + k$; to the left it is $y = -a(x - h) + k$.



$y = a|x - h| + k$, with $a \neq 0$:

- vertex (h, k); right arm slope a, left arm slope $-a$;
- one step either side of the vertex, the height changes by a;
- $a > 0$: opens up, range $y \geq k$. $a < 0$: opens down, range $y \leq k$;
- $|a| > 1$: narrower than $|x|$. $|a| < 1$: wider. The domain is all real numbers.

2. Take a factor out of the bars first

Since $|mn| = |m|\,|n|$, a number multiplying x inside the bars can come out. For example $|5x - 10| = |5(x - 2)| = 5|x - 2|$. Only then read h.

3. Rules and equations from the graph

The vertex gives h and k. Any other point (x, y) gives

$$a = \frac{y - k}{|x - h|}$$

To solve $a|x - h| + k = c$, find where the line $y = c$ meets the V: two points, only the vertex, or none. Two crossings are the same distance from h, one on each side. For an inequality, read where the V is above or below the line.

WATCH OUT

- **The sign inside flips.** $|x + 5|$ is $|x - (-5)|$, so $h = -5$.
- **Factor before you read the vertex.** In $|2x - 14|$ the corner is at $x = 7$, not $x = 14$.
- **The two arms have opposite slopes.** If the slopes of two pieces are not opposites, the graph is not a V of this family.
- **A limited domain cuts the range.** If x only runs over a stretch, the range is only the heights reached on that stretch.

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WORKED EXAMPLES

Example 1. Graph $y = -2|x - 5| + 6$. Give the vertex, the slopes of the arms, the range and the intercepts.

1. Match with $a|x - h| + k$: $a = -2$, $h = 5$, $k = 6$. The vertex is $(5, 6)$.
2. Since $a < 0$ the V opens down, the vertex is the highest point, and the range is $y \le 6$. The right arm has slope -2, the left arm slope 2.
3. One step either side the height drops 2: $(4, 4)$ and $(6, 4)$.
4. y-intercept: at $x = 0$, $y = -2|-5| + 6 = -10 + 6 = -4$.
5. x-intercepts: $-2|x - 5| + 6 = 0$ gives $|x - 5| = 3$, so $x = 2$ or $x = 8$.

Answer: vertex $(5, 6)$; slopes 2 (left) and -2 (right); range $y \le 6$; intercepts $(0, -4)$, $(2, 0)$ and $(8, 0)$. **Check:** $-2|8 - 5| + 6 = -6 + 6 = 0$.

Example 2. Find the vertex, the slopes of the arms and the range of $y = |3x - 6| - 4$.

1. Take the factor out: $3x - 6 = 3(x - 2)$, so $|3x - 6| = 3|x - 2|$.
2. The rule is $y = 3|x - 2| - 4$: $a = 3$, $h = 2$, $k = -4$.
3. The vertex is $(2, -4)$. The left arm has slope -3 and the right arm slope 3. Since $a > 0$, the range is $y \ge -4$.

Answer: vertex $(2, -4)$; slopes -3 and 3; range $y \ge -4$. **Check:** at $x = 0$, the first rule gives $|-6| - 4 = 2$ and the new one gives $3 \cdot 2 - 4 = 2$.

Example 3. A V has its vertex at $(-1, -5)$ and passes through $(2, 1)$. Write its rule, then solve $y \le 3$.

1. The vertex gives $h = -1$ and $k = -5$, so $y = a|x + 1| - 5$.
2. Use $(2, 1)$: $a = \dfrac{1 - (-5)}{|2 - (-1)|} = \dfrac{6}{3} = 2$.
3. The V opens up, so it is on or below the line $y = 3$ between the two crossings.
4. Crossings: $2|x + 1| - 5 = 3$ gives $|x + 1| = 4$, so $x = -5$ or $x = 3$.

Answer: $y = 2|x + 1| - 5$, and $y \le 3$ for $-5 \le x \le 3$. **Check:** the point $(-4, 1)$, one the same distance on the other side, fits: $2|-4 + 1| - 5 = 6 - 5 = 1$.

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