Stretching and compressing a graph

A number that multiplies the output pulls a graph away from the x-axis or squashes it toward it. A number that multiplies the input does the same sideways, but backwards. How do you tell which, and by how much?

WORDS TO KNOW

Vertical stretch: every point moves straight away from the x-axis; its height is multiplied by a number bigger than 1.

Vertical compression: every point moves straight toward the x-axis; its height is multiplied by a number between 0 and 1.

Horizontal stretch and **compression:** the same, but sideways, away from or toward the y-axis.

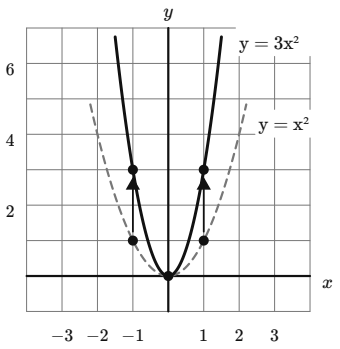
Factor: the number each distance is multiplied by.

Reciprocal: the reciprocal of b is $\frac{1}{b}$. The reciprocal of 4 is $\frac{1}{4}$.

KEY IDEAS

1. A number times the output: $a\,f(x)$

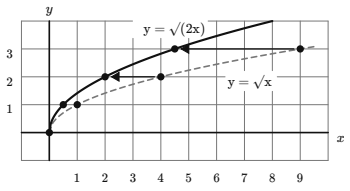
With $a > 0$, $g(x) = a\,f(x)$ works out f and then multiplies by a. Every point keeps its input, and its height is multiplied by a.



> $y = a\,f(x)$: $(p, q) \mapsto (p, aq)$.
>
> $a > 1$: stretched vertically by a factor of a. $0 < a < 1$: compressed vertically by a factor of a.
>
> Points on the x-axis stay. The domain does not change. A negative a also flips the graph across the x-axis.

2. A number times the input: $f(bx)$, read backwards

With $b > 0$, $g(x) = f(bx)$ multiplies the input by b before f works. So g gives the output q where $bx = p$, that is at $x = \frac{p}{b}$. The distance from the y-axis is multiplied by $\frac{1}{b}$, the reciprocal.



> $y = f(bx)$: $(p, q) \mapsto \left(\frac{p}{b}, q\right)$.
>
> $b > 1$: compressed horizontally by a factor of $\frac{1}{b}$. $0 < b < 1$: stretched horizontally by a factor of $\frac{1}{b}$.
>
> Points on the y-axis stay. The range does not change.

3. Naming a stretch from points

For a vertical move, compare **outputs at the same input**: $a = \frac{g(p)}{f(p)}$. For a horizontal move, compare **inputs at the same output**. One pair of points gives the number; a second pair confirms it.

For x^2, x^3, $\sqrt{x}$ and $\frac{1}{x}$, every horizontal move is also a vertical move, so one graph can have two correct names.

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WATCH OUT

- **A big number inside squeezes.** $f(3x)$ compresses the graph to a third of its width, even though 3 > 1. Each height comes at a third of the input.
- **"Away from the axis", not "up".** In a vertical stretch a point below the x-axis moves down, further from the axis.
- **Compare at the same input, or the same output.** For a vertical factor, divide the two heights at one x. Comparing points with different inputs gives a wrong factor.
- **The number cannot be 0.** $0 \cdot f(x)$ squashes every point onto the x-axis. That is not a stretch, and it cannot be undone.

WORKED EXAMPLES

Example 1. Describe $g(x) = 2x^3$ and $k(x) = \frac{1}{6}\sqrt{x}$ as moves of their parents. Give the images of (2, 8) and (−1, −1) for g, and of (36, 6) for k.

1. In g, the 2 multiplies the whole output of x^3. So every height doubles and every input stays: a vertical stretch by a factor of 2.
2. (2, 8) goes to (2, 16). (−1, −1) goes to (−1, −2): below the axis, it moves down, away from the axis.
3. In k, the $\frac{1}{6}$ multiplies the output of $\sqrt{x}$: a vertical compression by a factor of $\frac{1}{6}$. (36, 6) goes to (36, 1).

Answer: g is $y = x^3$ stretched vertically by 2, with images $(2, 16)$ and $(-1, -2)$. k is $y = \sqrt{x}$ compressed vertically by $\frac{1}{6}$, with image $(36, 1)$. **Check:** $2 \cdot 2^3 = 16$, $2 \cdot (-1)^3 = -2$, and $\frac{1}{6}\sqrt{36} = 1$.

Example 2. Describe $p(x) = \sqrt{25x}$ as a horizontal move of $y = \sqrt{x}$. Then name it as a vertical move too.

1. The 25 multiplies the input, so expect the reciprocal: each point moves to $\frac{1}{25}$ of its distance from the y-axis.
2. The parent reaches height 5 at $x = 25$. p reaches it at $x = 1$, since $\sqrt{25 \cdot 1} = 5$. The parent reaches 10 at $x = 100$, and p at $x = 4$.
3. So p is $y = \sqrt{x}$ compressed horizontally by a factor of $\frac{1}{25}$.
4. Simplify: $\sqrt{25x} = \sqrt{25} \cdot \sqrt{x} = 5\sqrt{x}$. That is the parent stretched vertically by a factor of 5.

Answer: compressed horizontally by a factor of $\frac{1}{25}$; also stretched vertically by a factor of 5. **Check:** at $x = 9$, $\sqrt{225} = 15$ and $5\sqrt{9} = 15$.

Example 3. A parabola with its vertex at the origin is a vertical move of $y = x^2$. It passes through (4, 10) and (−2, 2.5). Name the move and write the rule.

1. Compare outputs at the same input. At $x = 4$ the parent gives 16 and this graph gives 10.
2. So $a = \frac{10}{16} = \frac{5}{8}$. This is less than 1, so it is a compression.
3. Confirm with the second point. At $x = -2$ the parent gives 4, and $\frac{5}{8} \cdot 4 = 2.5$.

Answer: $y = \frac{5}{8}x^2$, the parabola $y = x^2$ compressed vertically by a factor of $\frac{5}{8}$. **Check:** $\frac{5}{8} \cdot 4^2 = \frac{5}{8} \cdot 16 = 10$.

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