Algebra 2 Unit 1 worksheet: The square root and reciprocal functions (answer key)

ANSWERS AT A GLANCE

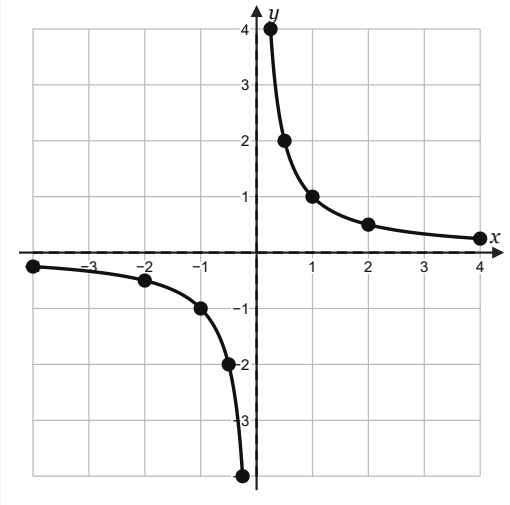
1. Points $(\frac{1}{4}, 4)$, $(\frac{1}{2}, 2)$, $(1, 1)$, $(2, \frac{1}{2})$, $(4, \frac{1}{4})$ and $(-\frac{1}{4}, -4)$, $(-\frac{1}{2}, -2)$, $(-1, -1)$, $(-2, -\frac{1}{2})$, $(-4, -\frac{1}{4})$; dashed asymptotes $x = 0$ and $y = 0$; two separate pieces, in the first and third quadrants.
2. (a) $\frac{1}{4} \le y \le 1$ (b) $-\frac{1}{2} \le y \le -\frac{1}{5}$ (c) $0 < y < \frac{1}{8}$ (d) $y > 5$ (e) $y \le -1$ or $y \ge \frac{1}{2}$
3. (a) $(0.625, 1.6)$, $(-1.6, -0.625)$ and $(-0.625, -1.6)$. (b) Each product is 1. (c) The mirror image of $(1.6, 0.625)$ in the y-axis is $(-1.6, 0.625)$, whose coordinates multiply to -1, so it is not on the graph.
4. $\left(\frac{1}{4}, 4\right)$ and $\left(-\frac{1}{4}, -4\right)$. The line $y = -16x$ never meets the graph, because for every $x \neq 0$ its output and $\frac{1}{x}$ have opposite signs.
5. (a) At $x = 1$ the square root is higher (1 against 0.25); at $x = 100$ the line is higher (25 against 10). (b) $x = 0$ and $x = 16$. (c) $0 < x < 16$.
6. (a) The area is always 1. (b) 8.5 (c) $a = 2$, with $P = (2, \frac{1}{2})$, and $a = \frac{1}{2}$, with $P = (\frac{1}{2}, 2)$.

WORKED SOLUTIONS

1. On graph paper, with x and y each running from -4 to 4, sketch the graph of $y = \frac{1}{x}$. Plot the exact points at $x = \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1, \pm 2$ and ± 4, draw both asymptotes as dashed lines, and name the quadrants the graph lies in.

ANSWER

Points $(\frac{1}{4}, 4)$, $(\frac{1}{2}, 2)$, $(1, 1)$, $(2, \frac{1}{2})$, $(4, \frac{1}{4})$ and $(-\frac{1}{4}, -4)$, $(-\frac{1}{2}, -2)$, $(-1, -1)$, $(-2, -\frac{1}{2})$, $(-4, -\frac{1}{4})$; dashed asymptotes $x = 0$ and $y = 0$; two separate pieces, in the first and third quadrants.



SOLUTION

Each output is 1 divided by the input, so the coordinates of every point multiply to 1:

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

The asymptotes are the axes, $x = 0$ and $y = 0$, drawn dashed. The right piece runs through the five points with $x > 0$, climbing beside the y-axis at its left end and sinking toward the x-axis at its right end; the left piece is the same shape turned half a turn about the origin. The pieces are not joined: there is no point at $x = 0$.

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Grade the sketch on: all ten points correct; two separate pieces, each falling from left to right; each piece bending toward both axes without touching them; dashed asymptotes $x = 0$ and $y = 0$; first and third quadrants named.

2. Find the range of $y = \dfrac{1}{x}$ when the inputs are limited as shown.

(a) $1 \le x \le 4$ (b) $-5 \le x \le -2$ (c) $x > 8$ (d) $0 < x < 0.2$

(e) $-1 \le x \le 2,\ x \ne 0$

ANSWER

(a) $\dfrac{1}{4} \le y \le 1$ (b) $-\dfrac{1}{2} \le y \le -\dfrac{1}{5}$ (c) $0 < y < \dfrac{1}{8}$ (d) $y > 5$ (e) $y \le -1$ or $y \ge \dfrac{1}{2}$

SOLUTION

Each piece of the graph falls from left to right, so on a stretch of one piece the left end gives the largest output and the right end the smallest.

(a) $\dfrac{1}{1} = 1$ and $\dfrac{1}{4}$, so $\dfrac{1}{4} \le y \le 1$.

(b) At the left end $\dfrac{1}{-5} = -\dfrac{1}{5}$, the largest; at the right end $\dfrac{1}{-2} = -\dfrac{1}{2}$, the smallest. So $-\dfrac{1}{2} \le y \le -\dfrac{1}{5}$.

(c) At $x = 8$ the output would be $\dfrac{1}{8}$, but 8 is not included. Further right the outputs sink toward 0 and stay positive, so $0 < y < \dfrac{1}{8}$.

(d) At $x = 0.2$ the output would be 5, not included. As x shrinks toward 0 the outputs grow without bound, so $y > 5$.

(e) The inputs lie on both pieces. On $-1 \le x < 0$ the outputs start at −1 and fall without bound beside the y-axis: $y \le -1$. On $0 < x \le 2$ they come down from without bound to $\dfrac{1}{2}$: $y \ge \dfrac{1}{2}$. So $y \le -1$ or $y \ge \dfrac{1}{2}$. The outputs are not the stretch from $\dfrac{1}{-1}$ to $\dfrac{1}{2}$: the gap at $x = 0$ splits them.

3. The point (1.6, 0.625) is on the graph of $y = \dfrac{1}{x}$.

(a) Without dividing, use the two symmetries of the graph to name three more points on it.

(b) Check each of your points by multiplying its coordinates.

(c) Jin says the graph is also its own mirror image in the y-axis. Use the point (1.6, 0.625) to show that he is wrong.

ANSWER

(a) (0.625, 1.6), (−1.6, −0.625) and (−0.625, −1.6). (b) Each product is 1. (c) The mirror image of (1.6, 0.625) in the y-axis is (−1.6, 0.625), whose coordinates multiply to −1, so it is not on the graph.

SOLUTION

(a) The graph is its own mirror image in the line $y = x$, which swaps the coordinates: (0.625, 1.6). It is also its own half turn about the origin, which changes both signs: (−1.6, −0.625). Doing both gives (−0.625, −1.6).

(b) 1.6 × 0.625 = 1, and changing both signs or swapping the order leaves the product 1, so all three points satisfy $xy = 1$.

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(c) A mirror in the y-axis changes only the sign of x, giving $(-1.6, 0.625)$. Its coordinates multiply to -1, not 1; in fact $\frac{1}{-1.6} = -0.625$. So the mirror image is not on the graph, and the graph is not symmetric about the y-axis.

4. Find every point where the line $y = 16x$ meets the graph of $y = \frac{1}{x}$. Then explain why the line $y = -16x$ never meets that graph.

ANSWER

$\left(\frac{1}{4}, 4\right)$ and $\left(-\frac{1}{4}, -4\right)$. The line $y = -16x$ never meets the graph, because for every $x \neq 0$ its output and $\frac{1}{x}$ have opposite signs.

SOLUTION

Where they meet, $\frac{1}{x} = 16x$ with $x \neq 0$. Multiplying by x gives $1 = 16x^2$, so $x^2 = \frac{1}{16}$ and $x = \frac{1}{4}$ or $x = -\frac{1}{4}$.

At $x = \frac{1}{4}$, $y = \frac{1}{1/4} = 4$; at $x = -\frac{1}{4}$, $y = -4$. The points are $\left(\frac{1}{4}, 4\right)$ and $\left(-\frac{1}{4}, -4\right)$, one on each piece, each the half turn of the other about the origin. Check: $16 \cdot \frac{1}{4} = 4$.

For $y = -16x$: when $x > 0$, $\frac{1}{x}$ is positive and $-16x$ is negative; when $x < 0$ it is the other way round. They are never equal. In algebra, $\frac{1}{x} = -16x$ would need $x^2 = -\frac{1}{16}$, and no square is negative. The line passes through the second and fourth quadrants, where the graph has no points.

5. Compare $y = \sqrt{x}$ with the line $y = \frac{x}{4}$ for $x \geq 0$.

(a) Which graph is higher at $x = 1$? At $x = 100$?

(b) Find every x at which the two graphs meet.

(c) Find every x at which the square root graph is above the line, and explain your answer.

ANSWER

(a) At $x = 1$ the square root is higher (1 against 0.25); at $x = 100$ the line is higher (25 against 10). (b) $x = 0$ and $x = 16$. (c) $0 < x < 16$.

SOLUTION

(a) $\sqrt{1} = 1$ and $\frac{1}{4} = 0.25$, so the square root graph is higher at $x = 1$. $\sqrt{100} = 10$ and $\frac{100}{4} = 25$, so the line is higher at $x = 100$.

(b) Squaring $\sqrt{x} = \frac{x}{4}$ gives $x = \frac{x^2}{16}$, so $x^2 - 16x = 0$ and $x(x - 16) = 0$: $x = 0$ or $x = 16$. Both check: $\sqrt{0} = 0 = \frac{0}{4}$ and $\sqrt{16} = 4 = \frac{16}{4}$.

(c) At $x = 0$ they are equal. For $x > 0$, $\sqrt{x}$ is positive, so dividing $\sqrt{x} > \frac{x}{4}$ by $\sqrt{x}$ keeps the direction. Since $x = \sqrt{x} \cdot \sqrt{x}$, this gives $1 > \frac{\sqrt{x}}{4}$, that is $\sqrt{x} < 4$. The square root rises, and $\sqrt{16} = 4$, so this holds exactly when $x < 16$. The square root graph is above the line for $0 < x < 16$, and below it for $x > 16$: the square root keeps rising, but more and more slowly, while the line rises at a steady rate.

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6. For a number $a > 0$, the point $P = \left(a, \frac{1}{a}\right)$ is on the graph of $y = \frac{1}{x}$. A rectangle has corners at the origin, at $(a, 0)$, at P and at $\left(0, \frac{1}{a}\right)$.

(a) Show that the area of the rectangle is the same for every $a > 0$.

(b) Find the perimeter of the rectangle when $a = 4$.

(c) Find every a that makes the perimeter 5, and name the point P in each case.

ANSWER

(a) The area is always 1. (b) 8.5 (c) $a = 2$, with $P = (2, \frac{1}{2})$, and $a = \frac{1}{2}$, with $P = (\frac{1}{2}, 2)$.

SOLUTION

(a) The rectangle is a wide and $\frac{1}{a}$ tall, so its area is $a \cdot \frac{1}{a} = 1$, whatever a is. This is the fact that the coordinates of every point on the graph multiply to 1.

(b) When $a = 4$ the sides are 4 and $\frac{1}{4}$, so the perimeter is $2\left(4 + \frac{1}{4}\right) = 8.5$.

(c) The perimeter is $2\left(a + \frac{1}{a}\right) = 5$, so $a + \frac{1}{a} = \frac{5}{2}$. Multiplying by $2a$, allowed since $a > 0$, gives $2a^2 + 2 = 5a$, so $2a^2 - 5a + 2 = 0$ and $(2a - 1)(a - 2) = 0$. Then $a = \frac{1}{2}$ or $a = 2$.

The points are $P = (2, \frac{1}{2})$ and $P = (\frac{1}{2}, 2)$. They are mirror images in the line $y = x$, so the two rectangles are the same shape turned on its side, with the same perimeter. Check: $2\left(2 + \frac{1}{2}\right) = 5$.

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