Algebra 2 Unit 1 worksheet: Reflecting a graph across the axes (answer key)

ANSWERS AT A GLANCE

1. (a) $g(x) = f(-x)$ (b) $g(x) = -f(x)$ (c) neither (d) both
2. (a) Domain $-3 \le x \le 4$; x-intercepts (−3, 0), (1, 0), (4, 0); y-intercept (0, 3).
   (b) (3, 0), (0, 3), (−1, 0), (−3, −3) and (−4, 0).
   (c) Domain $-4 \le x \le 3$; x-intercepts (−4, 0), (−1, 0), (3, 0); y-intercept (0, 3).
   (d) $-4 < x < -1$
3. (a) $f(1) = -2$, so (1, −2) should go to (−1, −2); Aisha's rule gives −4 at $x = -1$.
   (b) $f(-x) = x^2 + 4x + 1$.
   (c) f: vertex (2, −3); $f(-x)$: vertex (−2, −3). Both are lowest points, and both ranges are $y \ge -3$.
4. (a) $-\sqrt{x+3} - 2$, then $-\sqrt{3-x} - 2$. (b) $\sqrt{3-x} + 2$, then $-\sqrt{3-x} - 2$: the same function. (c) Either order sends (a, b) to $(-a, -b)$. (d) Starts at (3, −2); domain $x \le 3$, range $y \le -2$.
5. (a) $r(x) = \frac{x^2}{4} - 4$ for $-4 \le x \le 4$. (b) Lowest point (0, −4); meets the water at (−4, 0) and (4, 0). (c) Arch $0 \le y \le 4$; reflection $-4 \le y \le 0$. (d) 6 meters.

WORKED SOLUTIONS

1. For each pair, decide whether $g(x) = -f(x)$, $g(x) = f(-x)$, both, or neither. Show the rewriting that decides it.

(a) $f(x) = x^2 - 6x,\ g(x) = x^2 + 6x$

(b) $f(x) = \dfrac{1}{x-4},\ g(x) = \dfrac{1}{4-x}$

(c) $f(x) = 2x^2 - 3,\ g(x) = -2x^2 - 3$

(d) $f(x) = x^3 - x,\ g(x) = x - x^3$

ANSWER

(a) $g(x) = f(-x)$ (b) $g(x) = -f(x)$ (c) neither (d) both

SOLUTION

Work out $-f(x)$ and $f(-x)$ for each and compare with $g(x)$.

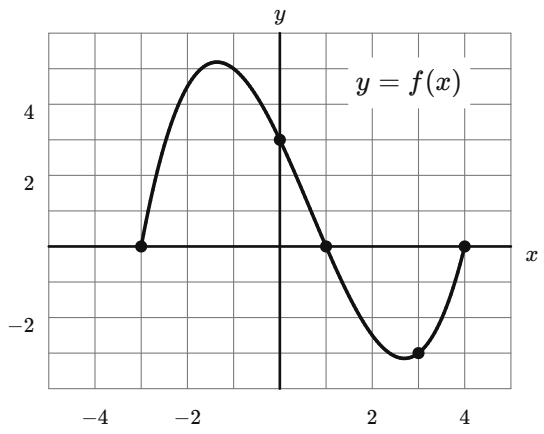
(a) $-f(x) = -x^2 + 6x$ and $f(-x) = (-x)^2 - 6(-x) = x^2 + 6x$. So $g(x) = f(-x)$ only.

(b) $-f(x) = -\dfrac{1}{x-4} = \dfrac{1}{-(x-4)} = \dfrac{1}{4-x}$ and $f(-x) = \dfrac{1}{-x-4}$, which is not the same (at $x = 0$ it is $-\dfrac{1}{4}$, while $g(0) = \dfrac{1}{4}$). So $g(x) = -f(x)$ only.

(c) $-f(x) = -(2x^2 - 3) = -2x^2 + 3$ and $f(-x) = 2(-x)^2 - 3 = 2x^2 - 3$. Neither is $-2x^2 - 3$: the minus sign in g reaches only the $2x^2$, not the whole output.

(d) $-f(x) = -x^3 + x$ and $f(-x) = (-x)^3 - (-x) = -x^3 + x$. Both equal $x - x^3$, so g is either reflection.

2. The figure shows the whole graph of f: a smooth curve with closed dots at its ends, through the marked points (−3, 0), (0, 3), (1, 0), (3, −3) and (4, 0).



(a) Give the domain of f, its x-intercepts and its y-intercept.

(b) Sketch the graph of $y = f(-x)$ on the same grid, and give the points that the five marked points go to.

(c) Give the domain, the x-intercepts and the y-intercept of $y = f(-x)$.

(d) For which inputs is $f(-x)$ negative?

ANSWER

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(a) Domain $-3 \le x \le 4$; x-intercepts (−3, 0), (1, 0), (4, 0); y-intercept (0, 3).

(b) (3, 0), (0, 3), (−1, 0), (−3, −3) and (−4, 0).

(c) Domain $-4 \le x \le 3$; x-intercepts (−4, 0), (−1, 0), (3, 0); y-intercept (0, 3).

(d) $-4 < x < -1$

SOLUTION

(a) The curve runs from $x = -3$ to $x = 4$, with closed dots at both ends, so the domain is $-3 \le x \le 4$. It meets the x-axis at (−3, 0), (1, 0) and (4, 0), and the y-axis at (0, 3).

(b) $f(-x)$ sends each point (a, b) to $(-a, b)$: each point moves straight across the y-axis to the same height. The marked points go to (3, 0), (0, 3), (−1, 0), (−3, −3) and (−4, 0). The sketch is the curve seen in a mirror along the y-axis: it starts at (−4, 0), dips to a low point a little below −3 near $x = -2.7$, rises through (−1, 0) and (0, 3) to a peak a little above 5 near $x = 1.4$, and comes down to (3, 0).

(c) The inputs change sign, so $-3 \le x \le 4$ becomes $-4 \le x \le 3$. The x-intercepts move across to (−4, 0), (−1, 0) and (3, 0). The y-intercept is on the mirror, so it stays at (0, 3).

(d) f is negative between its zeros 1 and 4, that is for $1 < x < 4$. Then $f(-x) < 0$ exactly when $1 < -x < 4$, which is $-4 < x < -1$: the part of the graph below the axis has moved across too.

3. Aisha reflects the graph of $f(x) = x^2 - 4x + 1$ across the y-axis. She changes the sign of every term that has an x in it and writes $f(-x) = -x^2 + 4x + 1$.

(a) The point $(1, f(1))$ is on the graph of f. Find it, say where the reflection across the y-axis should send it, and show that Aisha's rule does not pass through that point.

(b) Write $f(-x)$ correctly and simplify it.

(c) Give the vertex of each graph, $y = f(x)$ and $y = f(-x)$, and say whether it is a highest or a lowest point. Give the range of each.

ANSWER

(a) $f(1) = -2$, so (1, −2) should go to (−1, −2); Aisha's rule gives −4 at $x = -1$.

(b) $f(-x) = x^2 + 4x + 1$.

(c) f: vertex (2, −3); $f(-x)$: vertex (−2, −3). Both are lowest points, and both ranges are $y \ge -3$.

SOLUTION

(a) $f(1) = 1 - 4 + 1 = -2$, so (1, −2) is on the graph of f. A reflection across the y-axis keeps the output and changes the sign of the input, so (−1, −2) should be on the reflected graph. Aisha's rule at $x = -1$ gives $-(-1)^2 + 4(-1) + 1 = -1 - 4 + 1 = -4$, not −2.

(b) Replace every x by $-x$, inside the square too: $f(-x) = (-x)^2 - 4(-x) + 1 = x^2 + 4x + 1$. The square of $-x$ is x^2, so the first term keeps its sign; Aisha negated it.

(c) Complete the square: $f(x) = (x - 2)^2 - 3$, so the vertex is (2, −3), a lowest point since the parabola opens upward, and the range is $y \ge -3$. Likewise $f(-x) = (x + 2)^2 - 3$, with vertex (−2, −3), the mirror image of (2, −3). A reflection across the y-axis does not change outputs, so this parabola still opens upward, its vertex is still a lowest point, and its range is still $y \ge -3$.

4. Start with $f(x) = \sqrt{x + 3} + 2$.

(a) Reflect its graph across the x-axis, then reflect the result across the y-axis. Write the rule after each step.

(b) Do the two reflections in the other order, writing the rule after each step. Do you reach the same function?

(c) Explain, by following a point (a, b) of the graph of f, why the order of the two reflections never matters.

(d) Give the starting point, the domain and the range of the final graph.

ANSWER

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(a) $-\sqrt{x+3}-2$, then $-\sqrt{3-x}-2$. (b) $\sqrt{3-x}+2$, then $-\sqrt{3-x}-2$: the same function. (c) Either order sends (a, b) to $(-a, -b)$. (d) Starts at (3, −2); domain $x \le 3$, range $y \le -2$.

SOLUTION

(a) Across the x-axis the minus sign reaches the whole output: $g(x) = -f(x) = -\left(\sqrt{x+3}+2\right) = -\sqrt{x+3}-2$. Then across the y-axis, replacing every x by $-x$: $g(-x) = -\sqrt{-x+3}-2 = -\sqrt{3-x}-2$.

(b) Across the y-axis first: $h(x) = f(-x) = \sqrt{-x+3}+2 = \sqrt{3-x}+2$. Then across the x-axis: $-h(x) = -\sqrt{3-x}-2$. Both orders reach $-\sqrt{3-x}-2$.

(c) The flip across the x-axis changes only the sign of the output, and the flip across the y-axis only the sign of the input. In one order $(a, b) \mapsto (a, -b) \mapsto (-a, -b)$; in the other $(a, b) \mapsto (-a, b) \mapsto (-a, -b)$. Each flip works on a different coordinate, so neither undoes or changes the other's work, and every point ends at $(-a, -b)$ either way.

(d) The graph of f starts at (−3, 2), which goes to (3, −2). The square root needs $3 - x \ge 0$, so $x \le 3$. The square root is never negative, so $-\sqrt{3-x} \le 0$ and $-\sqrt{3-x} - 2 \le -2$: the range is $y \le -2$. The curve starts at (3, −2) and runs to the left, falling.

5. A stone footbridge crosses a canal. The underside of its arch follows $y = 4 - \dfrac{x^2}{4}$ for $-4 \le x \le 4$, where x is meters from the middle of the canal and y is meters above the water. On a still day the water shows the arch upside down, as its reflection across the water line, the x-axis.

(a) Write a rule $r(x)$ for the reflected arch, simplified.

(b) Give the lowest point of the reflected arch, and the points where it meets the water line.

(c) Give the range of the real arch and the range of the reflected arch.

(d) The arch and its reflection together look like a closed ring. How tall is the ring, from its top edge to its bottom edge, at 2 meters from the middle?

ANSWER

(a) $r(x) = \dfrac{x^2}{4} - 4$ for $-4 \le x \le 4$. (b) Lowest point (0, −4); meets the water at (−4, 0) and (4, 0). (c) Arch $0 \le y \le 4$; reflection $-4 \le y \le 0$. (d) 6 meters.

SOLUTION

(a) Reflecting across the x-axis changes the sign of every output: $r(x) = -\left(4 - \dfrac{x^2}{4}\right) = \dfrac{x^2}{4} - 4$, with the same domain $-4 \le x \le 4$.

(b) The top of the real arch is (0, 4), its highest point. The reflection sends it to (0, −4), which becomes the lowest point of the reflected arch, 4 m below the water line. The real arch meets the water where $4 - \dfrac{x^2}{4} = 0$, so $x^2 = 16$ and $x = \pm 4$. Those points lie on the mirror and do not move: (−4, 0) and (4, 0).

(c) On $-4 \le x \le 4$ the real arch runs from 0 at the ends up to 4 in the middle: range $0 \le y \le 4$. Every output changes sign, so the reflection has range $-4 \le y \le 0$.

(d) At $x = 2$ the arch is at $4 - \dfrac{4}{4} = 3$ m and the reflection at $r(2) = \dfrac{4}{4} - 4 = -3$ m. The ring is $3 - (-3) = 6$ m tall there, twice the height of the arch.

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