Algebra 2 Unit 1 worksheet: The squaring and cubing functions (answer key)

ANSWERS AT A GLANCE

1. $y = x^2$: (a) all real numbers (b) $y \geq 0$ (c) falls for $x \leq 0$, rises for $x \geq 0$ (d) the vertex $(0, 0)$ (e) two inputs give 5; no input gives −5.
   $y = x^3$: (a) all real numbers (b) all real numbers (c) rises for every x, falls nowhere (d) none (e) one input gives 5, and one gives −5.
2. $(-4, 16)$ on $y = x^2$, by its mirror symmetry in the y-axis; $(-3, -27)$ on $y = x^3$, by its half-turn symmetry about the origin.
3. (a) The U passes through $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(2, 4)$; the V through $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(2, 2)$.
   (b) $(-1, 1)$, $(0, 0)$ and $(1, 1)$
   (c) $-1 < x < 0$ and $0 < x < 1$
4. $a = -4$. The right end gives only $3^2 = 9$, which is less than 16.
5. (a) $b = -a$: the inputs are opposites, and neither is 0. (b) No. The cubic rises everywhere, so different inputs always give different outputs.
6. Comparing the two ends does not show what happens between them. The graph falls from $(-4, 16)$ to the vertex $(0, 0)$, then rises to $(1, 1)$. The range there is $0 \leq y \leq 16$.
7. (a) 8 cm^3 and 64 cm^3 (b) 8 times as much ice; a face is 4 times as large. (c) A face's area squares the edge and the volume cubes it, so multiplying the edge by 2 multiplies the area by 2^2 and the volume by 2^3.

WORKED SOLUTIONS

1. For $y = x^2$, and then for $y = x^3$, give (a) the domain, (b) the range, (c) the inputs where the graph falls and the inputs where it rises, (d) the lowest point, if there is one, and (e) how many inputs give the output 5, and how many give the output −5.

> **ANSWER**
> $y = x^2$: (a) all real numbers (b) $y \geq 0$ (c) falls for $x \leq 0$, rises for $x \geq 0$ (d) the vertex $(0, 0)$ (e) two inputs give 5; no input gives −5.
>
> $y = x^3$: (a) all real numbers (b) all real numbers (c) rises for every x, falls nowhere (d) none (e) one input gives 5, and one gives −5.
>
> **SOLUTION**
> Any real number can be squared or cubed, so both domains are all real numbers.
>
> No square is negative, and every $c \geq 0$ is the square of $\sqrt{c}$, so the range of $y = x^2$ is $y \geq 0$. The U falls until $x = 0$ and rises after it, so its lowest point is the vertex $(0, 0)$. The height 5 is reached twice, at $x = \sqrt{5}$ and $x = -\sqrt{5}$, and the height −5 never.
>
> Cubing keeps the sign, and every real number has a cube root, so the range of $y = x^3$ is all real numbers. The cubic rises everywhere (it is level for an instant at the origin but does not turn), so it has no lowest point. It meets each horizontal line once: 5 at $x = \sqrt[3]{5}$ and −5 at $x = -\sqrt[3]{5}$.

2. The point $(4, 16)$ lies on the graph of $y = x^2$, and the point $(3, 27)$ lies on the graph of $y = x^3$. Without squaring or cubing anything, give the point of $y = x^2$ with input −4 and the point of $y = x^3$ with input −3. Name the symmetry you used for each.

> **ANSWER**
> $(-4, 16)$ on $y = x^2$, by its mirror symmetry in the y-axis; $(-3, -27)$ on $y = x^3$, by its half-turn symmetry about the origin.
>
> **SOLUTION**
> Squaring forgets the sign: $(-4)^2 = 4^2 = 16$. So the point with input −4 is at the same height, $(-4, 16)$, the mirror image of $(4, 16)$ in the y-axis.
>
> Cubing keeps the sign: $(-3)^3 = -(3^3) = -27$. So the point is $(-3, -27)$, where a half turn about the origin carries $(3, 27)$.

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3. (a) Sketch $y = x^2$ and the absolute value parent $y = |x|$ on one grid for $-2 \le x \le 2$, plotting five key points of each.

(b) Find every point where the two graphs meet.

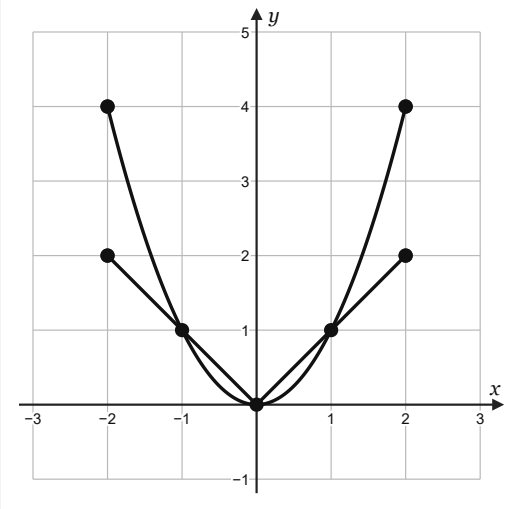
(c) For which inputs is the parabola below the V?

ANSWER

(a) The U passes through (−2, 4), (−1, 1), (0, 0), (1, 1), (2, 4); the V through (−2, 2), (−1, 1), (0, 0), (1, 1), (2, 2).

(b) (−1, 1), (0, 0) and (1, 1)

(c) $-1 < x < 0$ and $0 < x < 1$



SOLUTION

(a) Grade the sketch on: both graphs through the origin, which is the vertex of each; the U smooth and rounded at the bottom, the V with a sharp corner and straight arms of slope 1 and −1; the U at height 4 at $x = \pm 2$ and the V at height 2; both through $(\pm 1, 1)$; each its own mirror image in the y-axis.

(b) For $x \ge 0$, $|x| = x$, and $x^2 = x$ means $x(x - 1) = 0$, so $x = 0$ or $x = 1$. Both graphs are their own mirror images in the y-axis, so the meeting point (1, 1) has a partner (−1, 1). The graphs meet at (−1, 1), (0, 0) and (1, 1).

(c) For $0 < x < 1$, multiplying x by a number less than 1 makes it smaller, so $x^2 < x = |x|$: near the origin the U is flatter than the V. Past 1 the square is the larger. By the mirror symmetry the same holds for $-1 < x < 0$. At $x = 0$ and $x = \pm 1$ the two are equal, so those inputs are left out.

4. The inputs of $y = x^2$ are limited to $a \le x \le 3$, where a is a negative number, and the range is then $0 \le y \le 16$. Find a. Explain why the right end, $x = 3$, does not give the top of the range.

ANSWER

$a = -4$. The right end gives only $3^2 = 9$, which is less than 16.

SOLUTION

Since $a < 0 < 3$, the vertex input 0 is inside the stretch, so the lowest output is 0, as the range says.

The highest output comes from the end farther from 0. The right end gives $3^2 = 9$, less than 16, so the top of the range must come from the left end: $a^2 = 16$.

The two numbers whose square is 16 are 4 and −4, and a is negative, so $a = -4$.

Check: on $-4 \le x \le 3$ the left end is 4 from 0, farther than 3, and $(-4)^2 = 16$.

5. Leah is told only that two different inputs, a and b, give the same output of $y = x^2$.

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(a) What must be true of a and b? Explain using the graph.

(b) Can two different inputs give the same output of $y = x^3$? Explain.

ANSWER

(a) $b = -a$: the inputs are opposites, and neither is 0. (b) No. The cubic rises everywhere, so different inputs always give different outputs.

SOLUTION

(a) The output is a height c, and the horizontal line $y = c$ meets the U at $x = \sqrt{c}$ and $x = -\sqrt{c}$ only, when $c > 0$. The height 0 is reached once, at the vertex, so it cannot come from two different inputs. So a and b are $\sqrt{c}$ and $-\sqrt{c}$ in some order: $b = -a$, and $a \neq 0$. They are mirror partners in the y-axis.

(b) If $a < b$ then $a^3 < b^3$: the cubic rises everywhere, so it meets each horizontal line at most once. Two different inputs never share an output, which is why cubing can be undone and squaring cannot.

6. Ryan says the graph of $y = x^2$ falls the whole way across the inputs $-4 \leq x \leq 1$, because the output at $x = -4$ is 16 and the output at $x = 1$ is only 1. Explain what is wrong with his reasoning, say what the graph really does on this stretch, and give its range there.

ANSWER

Comparing the two ends does not show what happens between them. The graph falls from (−4, 16) to the vertex (0, 0), then rises to (1, 1). The range there is $0 \leq y \leq 16$.

SOLUTION

Two outputs at the ends only say which end is higher. The squaring graph falls for $x \leq 0$ and rises for $x \geq 0$, and the input 0 lies inside the stretch. So the graph falls from (−4, 16) down to the vertex (0, 0), then rises to (1, 1): it does some of each.

The lowest output is the vertex height 0, not the 1 at the right end. The highest comes from the input farther from 0, which is −4, with output 16. The graph is unbroken, so every height between is reached, and the range is $0 \leq y \leq 16$.

7. Mia's ice tray makes cubes with edges 2 cm long. Her new tray makes cubes with edges 2 times as long. (a) Find the volume of one old cube and of one new cube. (b) How many times as much ice is in one new cube as in one old cube? How many times as large is one face of a new cube as one face of an old cube? (c) Explain why the two answers to (b) differ, using the rules $y = x^2$ and $y = x^3$.

ANSWER

(a) 8 cm^3 and 64 cm^3 (b) 8 times as much ice; a face is 4 times as large. (c) A face's area squares the edge and the volume cubes it, so multiplying the edge by 2 multiplies the area by 2^2 and the volume by 2^3.

SOLUTION

(a) The old cube has volume $2^3 = 8$ cm^3. The new edge is 2 · 2 = 4 cm, so the new cube has volume $4^3 = 64$ cm^3.

(b) 64 ÷ 8 = 8, so a new cube holds 8 times as much ice. A face is a square: the old one has area 2^2 and the new one $4^2 = (2 \cdot 2)^2 = 2^2 \cdot 2^2$, so it is 4 times as large.

(c) Area follows the squaring rule and volume the cubing rule. Squaring 2 times the edge gives two factors of 2, and cubing gives three: $(2s)^2 = 4s^2$ and $(2s)^3 = 8s^3$. Each extra factor of 2, a number greater than 1, makes the result larger, so the volume grows faster than the area.

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