Algebra 2 Unit 1 worksheet: Even and odd functions (answer key)

ANSWERS AT A GLANCE

1. (a) even (b) neither (c) odd (d) neither (e) even (f) odd
2. (a) odd (b) odd (c) neither (d) even (e) neither
3. (a) the y-axis only (b) none (c) the origin only
4. (a) One input that works proves nothing: even means $f(-x) = f(x)$ for every x. (b) $f(-x) = x^4 - x^3 + 4x$; it equals $f(x)$ only at $x = -2$, 0 and 2. (c) No: $f(1) = -2$ and $f(-1) = 4$, so f is neither even nor odd.
5. (a) $k = 2$, giving $f(x) = x^4 + 5x^2$. (b) Neither: $f(x) = x^4 - 4x^3 + 5x^2$. (c) The terms x^4 and $5x^2$ keep their sign when x becomes $-x$, and their coefficients do not depend on k.
6. (a) $p(-x) = -p(x)$ and $q(-x) = q(x)$. (b) $p(-3) = 14$ and $q(-3) = 22$. (c) Oddness at $x = 0$ says $f(0) = -f(0)$.

WORKED SOLUTIONS

1. Decide whether each function is even, odd or neither. Simplify $f(-x)$ and compare it with $f(x)$ and $-f(x)$. When the answer is neither, give one input that shows it.

(a) $f(x) = 2x^6 - x^4 + 9$ (b) $f(x) = x^5 - 4x^3 + 2x^2$ (c) $f(x) = -3x^9 + x$

(d) $f(x) = x^4 - x^3$ (e) $f(x) = 7$ (f) $f(x) = (2x)^3 - x$

ANSWER

(a) even (b) neither (c) odd (d) neither (e) even (f) odd

SOLUTION

(a) Every power is even, and $9 = 9x^0$: $f(-x) = 2(-x)^6 - (-x)^4 + 9 = 2x^6 - x^4 + 9 = f(x)$. Even.

(b) $f(-x) = -x^5 + 4x^3 + 2x^2$. The odd-power terms changed sign but $2x^2$ did not, so this is neither $f(x)$ nor $-f(x) = -x^5 + 4x^3 - 2x^2$. With $x = 1$: $f(1) = -1$ and $f(-1) = 5$, neither equal nor opposite. Neither: the even-power term is what breaks oddness.

(c) Both powers are odd: $f(-x) = -3(-x)^9 + (-x) = 3x^9 - x = -f(x)$. Odd.

(d) $f(-x) = x^4 + x^3$, a mix of an even and an odd power. With $x = 1$: $f(1) = 0$ and $f(-1) = 2$, neither equal nor opposite. Neither.

(e) A constant is $7x^0$, an even power: $f(-x) = 7 = f(x)$. Even.

(f) Replace x by $-x$ inside the parentheses too: $f(-x) = (-2x)^3 - (-x) = -8x^3 + x = -(8x^3 - x) = -f(x)$. Odd.

2. Decide whether each function is even, odd or neither. Look at the domain first, then simplify $f(-x)$.

(a) $f(x) = \dfrac{x^2 + 2}{x^3}$ (b) $f(x) = \dfrac{|x|}{x}$ (c) $f(x) = \sqrt{16 - x^2} + x$

(d) $f(x) = |x^3 - 2x|$ (e) $f(x) = x^2$ for $-2 \le x \le 3$

ANSWER

(a) odd (b) odd (c) neither (d) even (e) neither

SOLUTION

(a) The domain is every $x \neq 0$. $f(-x) = \dfrac{(-x)^2 + 2}{(-x)^3} = \dfrac{x^2 + 2}{-x^3} = -f(x)$. Odd: an even function divided by an odd one.

(b) The domain is every $x \neq 0$. Since $|-x| = |x|$, $f(-x) = \dfrac{|x|}{-x} = -f(x)$. Odd. (The function is 1 for $x > 0$ and –1 for $x < 0$.)

(c) The domain is $-4 \le x \le 4$, symmetric. $f(-x) = \sqrt{16 - x^2} - x$, which is neither $f(x)$ nor $-f(x) = -\sqrt{16 - x^2} - x$. With $x = 0$: $f(0) = 4$, but an odd function would need $f(0) = -f(0)$, that is $f(0) = 0$; so

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f is not odd. With $x = 4$: $f(4) = 4$ and $f(-4) = -4$, so f is not even. Neither.

(d) $(-x)^3 - 2(-x) = -(x^3 - 2x)$, and a number and its opposite have the same absolute value, so $f(-x) = |-(x^3 - 2x)| = |x^3 - 2x| = f(x)$. Even.

(e) The domain $-2 \le x \le 3$ is not symmetric: $f(3) = 9$ exists but $f(-3)$ does not. Neither, even though the rule x^2 alone is even.

3. Test the graph of each equation for symmetry about the x-axis, the y-axis and the origin. For each symmetry it lacks, give a point on the graph whose partner is not on it.

(a) $x^2y = 6$ (b) $x = y^2 - 2y$ (c) $x^2 - xy + y^2 = 7$

ANSWER

(a) the y-axis only (b) none (c) the origin only

SOLUTION

(a) Replacing x by $-x$ gives $(-x)^2y = 6$, which is $x^2y = 6$ again: symmetric about the y-axis. Replacing y by $-y$, or both, gives $-x^2y = 6$, a different equation. The point (1, 6) is on the graph; its partner across the x-axis (1, −6) gives $1 \cdot (-6) = -6 \neq 6$, and its partner through the origin (−1, −6) also gives $-6 \neq 6$. The y-axis only.

(b) The point (3, 3) is on the graph, since $9 - 6 = 3$. Replacing y by $-y$ gives $x = y^2 + 2y$; the partner (3, −3) gives $9 + 6 = 15 \neq 3$. Replacing x by $-x$ gives $-x = y^2 - 2y$; the partner (−3, 3) gives $9 - 6 = 3 \neq -3$. Replacing both gives $-x = y^2 + 2y$; the partner (−3, −3) gives $15 \neq -3$. None. (It is a parabola opening to the right with its axis on the line $y = 1$, off the x-axis.)

(c) Replacing both: $(-x)^2 - (-x)(-y) + (-y)^2 = x^2 - xy + y^2$, the same equation, so the graph is symmetric about the origin. Replacing only one variable changes the sign of the xy term: $x^2 + xy + y^2 = 7$. The point (1, 3) is on the graph, since $1 - 3 + 9 = 7$. Its partner (1, −3) gives $1 + 3 + 9 = 13 \neq 7$, and its partner (−1, 3) also gives $13 \neq 7$. The origin only.

4. Arjun tests $f(x) = x^4 + x^3 - 4x$. He works out $f(2) = 16$ and $f(-2) = 16$, and writes: "$f(-2) = f(2)$, so f is even."

(a) Explain what is wrong with Arjun's reasoning.

(b) Simplify $f(-x)$ and use it to show that f is not even. Find every input x at which $f(-x) = f(x)$ does hold.

(c) Is f odd? Use $x = 1$ to decide.

ANSWER

(a) One input that works proves nothing: even means $f(-x) = f(x)$ for every x. (b) $f(-x) = x^4 - x^3 + 4x$; it equals $f(x)$ only at $x = -2$, 0 and 2. (c) No: $f(1) = -2$ and $f(-1) = 4$, so f is neither even nor odd.

SOLUTION

(a) "Even" is a statement about every input. Agreement at $x = 2$ checks one input only; it cannot rule out an input where $f(-x)$ and $f(x)$ differ. Showing evenness needs algebra on a general x.

(b) $f(-x) = (-x)^4 + (-x)^3 - 4(-x) = x^4 - x^3 + 4x$. Then

$$f(-x) - f(x) = -2x^3 + 8x = -2x(x^2 - 4) = -2x(x - 2)(x + 2),$$

which is not 0 for every x, so f is not even. It is 0 only at $x = -2$, 0 and 2: Arjun happened to test one of the three inputs where the two sides agree.

(c) $f(1) = 1 + 1 - 4 = -2$ and $f(-1) = 1 - 1 + 4 = 4$. Since $4 \neq -(-2) = 2$, $f(-1) \neq -f(1)$ and f is not odd. (Also $4 \neq -2$, which shows again that it is not even.) f is neither.

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5. Let $f(x) = x^4 + (k-2)x^3 + 5x^2 + (k^2-4)x$, where k is a number.

(a) Find every value of k for which f is even, and write f for that value.

(b) Decide whether f is even, odd or neither when $k = -2$.

(c) Explain why no value of k makes f odd.

ANSWER

(a) $k = 2$, giving $f(x) = x^4 + 5x^2$. (b) Neither: $f(x) = x^4 - 4x^3 + 5x^2$. (c) The terms x^4 and $5x^2$ keep their sign when x becomes $-x$, and their coefficients do not depend on k.

SOLUTION

(a) Replacing x by $-x$ keeps the even-power terms and flips the odd-power ones. So $f(-x) = f(x)$ for every x exactly when both odd-power coefficients are 0: $k - 2 = 0$ and $k^2 - 4 = 0$. The first needs $k = 2$, and $k = 2$ also gives $k^2 - 4 = 0$. (The value $k = -2$ clears only the second.) So $k = 2$ and $f(x) = x^4 + 5x^2$.

(b) With $k = -2$: $f(x) = x^4 - 4x^3 + 5x^2$, and $f(-x) = x^4 + 4x^3 + 5x^2$. With $x = 1$: $f(1) = 2$ and $f(-1) = 10$, neither equal nor opposite. Neither.

(c) For f to be odd, every term must change sign when x becomes $-x$, so every even-power coefficient must be 0. The coefficients of x^4 and x^2 are 1 and 5 whatever k is. For example $f(1) + f(-1) = 2 + 10 = 12$ for every k, while an odd function has $f(1) + f(-1) = 0$.

6. The function f is odd and the function g is even, both defined for every real number. All you know about their values is that $f(3) = 4$ and $g(3) = -2$. Let

$$p(x) = f(x)\,g(x) + x\,g(x), \qquad q(x) = f(x)^2 - 3g(x).$$

(a) Show that p is odd and q is even.

(b) Find $p(-3)$ and $q(-3)$.

(c) Explain why $f(0)$ must be 0.

ANSWER

(a) $p(-x) = -p(x)$ and $q(-x) = q(x)$. (b) $p(-3) = 14$ and $q(-3) = 22$. (c) Oddness at $x = 0$ says $f(0) = -f(0)$.

SOLUTION

(a) Use $f(-x) = -f(x)$ and $g(-x) = g(x)$:

$$p(-x) = \big(-f(x)\big)\,g(x) + (-x)\,g(x) = -\big(f(x)\,g(x) + x\,g(x)\big) = -p(x),$$

so p is odd (both terms are odd times even). And $q(-x) = \big(-f(x)\big)^2 - 3g(x) = f(x)^2 - 3g(x) = q(x)$, so q is even.

(b) $p(3) = 4(-2) + 3(-2) = -14$, so $p(-3) = -p(3) = 14$. Directly: $f(-3) = -4$ and $g(-3) = -2$, so $p(-3) = (-4)(-2) + (-3)(-2) = 8 + 6 = 14$. And $q(3) = 16 + 6 = 22$, so $q(-3) = q(3) = 22$.

(c) Put $x = 0$ into $f(-x) = -f(x)$: since −0 = 0, this says $f(0) = -f(0)$, so $2f(0) = 0$ and $f(0) = 0$.

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