Algebra 2 Unit 1 worksheet: Stretches and compressions (answer key)

ANSWERS AT A GLANCE

1. (a) $y = \frac{1}{5}x^2$, compressed vertically by a factor of $\frac{1}{5}$ (b) $y = \frac{3}{4} \cdot \frac{1}{x}$, compressed vertically by a factor of $\frac{3}{4}$ (c) $y = 4\sqrt{x}$, stretched vertically by a factor of 4 (d) $y = \frac{1}{27}x^3$, compressed vertically by a factor of $\frac{1}{27}$
2. $a = \frac{2}{5}$. $g(-4) = \frac{12}{5}$, $f(-1) = 5$, $g(0) = -\frac{6}{5}$, $f(2) = -15$. The graph of f is compressed vertically by a factor of $\frac{2}{5}$.
3. (a) $a = \frac{1}{3}$: $y = x^2$ compressed vertically by a factor of $\frac{1}{3}$. (b) 36 units
4. (a) g: the graph of f compressed horizontally by a factor of $\frac{1}{3}$. k: the graph of f stretched horizontally by a factor of 2. (b) On g: $(-2, 3)$, $(0, -2)$, $(2, 0)$, $(4, 5)$. On k: $(-12, 3)$, $(0, -2)$, $(12, 0)$, $(24, 5)$. The point $(0, -2)$ does not move: it is on the y-axis. (c) g: domain $-2 \le x \le 4$, range $-2 \le y \le 5$. k: domain $-12 \le x \le 24$, range $-2 \le y \le 5$.
5. (a) $g(-1) = -3 < -1$, so the claim fails. (b) $x > 0$. (c) Exactly when $f(x) > 0$: points above the axis move up, points on it stay, points below it move down.
6. (a) $f(x) = \frac{1}{4}g(x)$: compress vertically by a factor of $\frac{1}{4}$. (b) $y = 3\sqrt{x}$: $y = \sqrt{x}$ stretched vertically by a factor of 3. (c) A vertical compression by a factor of $\frac{1}{10}$.
7. (a) Segments joining $(-4, -3)$, $(-2, 3)$, $(0, 3)$, $(2, -3)$, $(4, 0)$. (b) Both have domain $-4 \le x \le 4$; f has range $-2 \le y \le 2$ and g has range $-3 \le y \le 3$. (c) The x-intercepts $(-3, 0)$, $(1, 0)$ and $(4, 0)$.
8. (a) A vertical stretch by a factor of 4. (b) $h(d)$: 0, 15, 35, -5, 0. (c) At $d = 0$ and $d = 800$, where the trail is level with the trailhead. (d) p: 0.4; h: 0.1. The drawing's climb is 4 times as steep.

WORKED SOLUTIONS

1. Each graph is a vertical stretch or compression of one of the parents $y = x^2$, $y = x^3$, $y = \sqrt{x}$ or $y = \frac{1}{x}$. Rewrite the rule in the form $y = a\,f(x)$, with f the parent, and name the move with its factor.

(a) $y = \frac{x^2}{5}$

(b) $y = \frac{3}{4x}$

(c) $y = \frac{\sqrt{x}}{0.25}$

(d) $y = \left(\frac{x}{3}\right)^3$

ANSWER

(a) $y = \frac{1}{5}x^2$, compressed vertically by a factor of $\frac{1}{5}$ (b) $y = \frac{3}{4} \cdot \frac{1}{x}$, compressed vertically by a factor of $\frac{3}{4}$ (c) $y = 4\sqrt{x}$, stretched vertically by a factor of 4 (d) $y = \frac{1}{27}x^3$, compressed vertically by a factor of $\frac{1}{27}$

SOLUTION

In each case pull the number that multiplies the whole output out to the front. A factor above 1 is a stretch, a factor between 0 and 1 a compression.

(a) Dividing by 5 is multiplying by $\frac{1}{5}$: $y = \frac{1}{5}x^2$. Parent $y = x^2$, compressed vertically by a factor of $\frac{1}{5}$.

(b) $\frac{3}{4x} = \frac{3}{4} \cdot \frac{1}{x}$. Parent $y = \frac{1}{x}$, compressed vertically by a factor of $\frac{3}{4}$.

(c) Dividing by $0.25 = \frac{1}{4}$ is multiplying by 4: $y = 4\sqrt{x}$. Parent $y = \sqrt{x}$, stretched vertically by a factor of 4. Dividing by a number between 0 and 1 stretches.

(d) $\left(\frac{x}{3}\right)^3 = \frac{x^3}{3^3} = \frac{1}{27}x^3$. Parent $y = x^3$, compressed vertically by a factor of $\frac{1}{27}$.

2. $g(x) = a\,f(x)$ for some number $a > 0$. Fill in the four blanks in the table, find a, and name the move from the graph of f to the graph of g.

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x	-4	-1	0	2	5
$f(x)$	6	_____	-3	_____	10
$g(x)$	_____	2	_____	-6	4

ANSWER

$a = \frac{2}{5}$. $g(-4) = \frac{12}{5}$, $f(-1) = 5$, $g(0) = -\frac{6}{5}$, $f(2) = -15$. The graph of f is compressed vertically by a factor of $\frac{2}{5}$.

SOLUTION

Only the column $x = 5$ gives both values, so it fixes a: $a = \frac{g(5)}{f(5)} = \frac{4}{10} = \frac{2}{5}$.

To get g from f, multiply by $\frac{2}{5}$: $g(-4) = \frac{2}{5} \cdot 6 = \frac{12}{5}$ and $g(0) = \frac{2}{5} \cdot (-3) = -\frac{6}{5}$.

To get f from g, divide by $\frac{2}{5}$, that is multiply by $\frac{5}{2}$: $f(-1) = \frac{5}{2} \cdot 2 = 5$ and $f(2) = \frac{5}{2} \cdot (-6) = -15$.

Since $0 < \frac{2}{5} < 1$, the graph of g is the graph of f compressed vertically by a factor of $\frac{2}{5}$.

3. The graph of $g(x) = ax^2$, with $a > 0$, is 18 units wide at height 27: its two points with $y = 27$ are 18 units apart.

(a) Find a and name the move from $y = x^2$.

(b) How wide is the graph of g at height 108?

ANSWER

(a) $a = \frac{1}{3}$: $y = x^2$ compressed vertically by a factor of $\frac{1}{3}$. (b) 36 units

SOLUTION

(a) Since $g(-x) = g(x)$, the two points at height 27 are equally far from the y-axis: they are (−9, 27) and (9, 27), half of 18 on each side. So $g(9) = 27$: $a \cdot 9^2 = 27$, $81a = 27$ and $a = \frac{1}{3}$. The graph is $y = x^2$ compressed vertically by a factor of $\frac{1}{3}$: $g(x) = \frac{1}{3}x^2$.

(b) Solve $\frac{1}{3}x^2 = 108$: $x^2 = 324$, so $x = 18$ or $x = -18$, and the width is 36 units. Four times the height gives twice the width, for any a: the square root of 4 is 2.

4. The graph of a function f is one unbroken curve with domain $-6 \le x \le 12$ and range $-2 \le y \le 5$. It passes through (−6, 3), (0, −2), (6, 0) and (12, 5). Let $g(x) = f(3x)$ and $k(x) = f\left(\frac{x}{2}\right)$.

(a) Name the move from the graph of f to the graph of g, and the move from the graph of f to the graph of k, each with its factor.

(b) Give the point of the graph of g and the point of the graph of k that each of the four points moves to. Which of the four points does not move, and why?

(c) Give the domain and range of g and of k.

ANSWER

(a) g: the graph of f compressed horizontally by a factor of $\frac{1}{3}$. k: the graph of f stretched horizontally by a factor of 2. (b) On g: (−2, 3), (0, −2), (2, 0), (4, 5). On k: (−12, 3), (0, −2), (12, 0), (24, 5). The point (0, −2) does not move: it is on the y-axis. (c) g: domain $-2 \le x \le 4$, range $-2 \le y \le 5$. k: domain $-12 \le x \le 24$, range $-2 \le y \le 5$.

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SOLUTION

(a) In $g(x) = f(3x)$ the input is multiplied by $b = 3$, so each point moves to $\frac{1}{3}$ of its distance from the y-axis: a horizontal compression by a factor of $\frac{1}{3}$. In $k(x) = f\left(\frac{x}{2}\right)$ the input is multiplied by $b = \frac{1}{2}$, so each point moves to $\frac{1}{b} = 2$ times its distance: a horizontal stretch by a factor of 2.

(b) g gives the output q when f receives p, that is when $3x = p$, at $x = \frac{p}{3}$. So $(p, q) \mapsto \left(\frac{p}{3}, q\right)$: $(-6, 3) \to (-2, 3)$, $(0, -2) \to (0, -2)$, $(6, 0) \to (2, 0)$, $(12, 5) \to (4, 5)$. For k, $\frac{x}{2} = p$ at $x = 2p$, so $(p, q) \mapsto (2p, q)$: $(-12, 3)$, $(0, -2)$, $(12, 0)$, $(24, 5)$. Check one: $k(24) = f(12) = 5$. The point $(0, -2)$ has input 0, and $\frac{0}{3} = 2 \cdot 0 = 0$, so a horizontal move leaves it in place.

(c) $g(x) = f(3x)$ is defined only when $-6 \le 3x \le 12$, that is $-2 \le x \le 4$. $k(x)$ is defined only when $-6 \le \frac{x}{2} \le 12$, that is $-12 \le x \le 24$. The outputs of g and k are outputs of f, and every output of f is reached, so both ranges stay $-2 \le y \le 5$.

5. Diego says: "$g(x) = 3x^3$ is $y = x^3$ stretched vertically by a factor of 3, so $g(x) > x^3$ for every x."

(a) Test Diego's claim at $x = -1$.

(b) Find every x for which $3x^3 > x^3$.

(c) Let f be any function and $a > 1$. Say exactly when $a\,f(x) > f(x)$, and describe what this means for points above, on and below the x-axis.

ANSWER

(a) $g(-1) = -3 < -1$, so the claim fails. (b) $x > 0$. (c) Exactly when $f(x) > 0$: points above the axis move up, points on it stay, points below it move down.

SOLUTION

(a) $g(-1) = 3(-1)^3 = -3$ and $(-1)^3 = -1$. Since $-3 < -1$, the stretched graph is lower there, and the claim is false.

(b) Subtract x^3: $3x^3 > x^3$ means $2x^3 > 0$, so $x^3 > 0$, which holds exactly when $x > 0$. At $x = 0$ the two are equal, and for $x < 0$ the stretch makes $3x^3$ smaller.

(c) $a\,f(x) > f(x)$ means $(a - 1)f(x) > 0$. Since $a - 1 > 0$, dividing by it keeps the sign: this holds exactly when $f(x) > 0$. A stretch pulls every point away from the x-axis: a point above the axis moves up (its output gets larger), a point on the axis stays (its output is 0 both times), and a point below the axis moves down (its output gets more negative, so smaller). "Stretched" means farther from the axis, not larger.

6. (a) $g(x) = 4f(x)$. Write $f(x)$ in terms of $g(x)$, and name the move that takes the graph of g back to the graph of f.

(b) Amara compresses $y = \sqrt{x}$ vertically by a factor of $\frac{1}{2}$, then stretches the new graph vertically by a factor of 6. Write the rule of her final graph and name it as one move of $y = \sqrt{x}$.

(c) A graph is stretched vertically by a factor of 5. What second vertical move makes the result the original graph compressed vertically by a factor of $\frac{1}{2}$?

ANSWER

(a) $f(x) = \frac{1}{4}g(x)$: compress vertically by a factor of $\frac{1}{4}$. (b) $y = 3\sqrt{x}$: $y = \sqrt{x}$ stretched vertically by a factor of 3. (c) A vertical compression by a factor of $\frac{1}{10}$.

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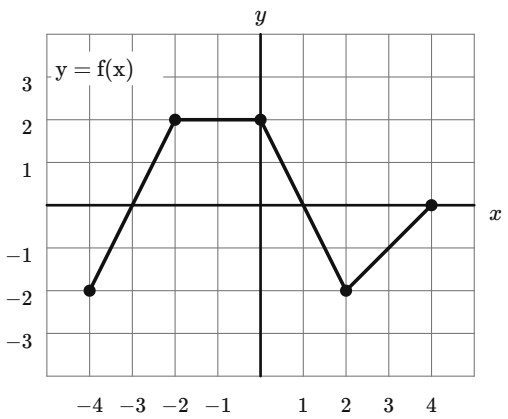
SOLUTION

(a) Divide both sides of $g(x) = 4f(x)$ by 4: $f(x) = \frac{1}{4}g(x)$. Each point of g goes back to a quarter of its distance from the x-axis, so the move from g back to f is a vertical compression by a factor of $\frac{1}{4}$. A stretch by a is undone by a compression by $\frac{1}{a}$.

(b) After the compression the rule is $y = \frac{1}{2}\sqrt{x}$. The stretch multiplies every output by 6: $y = 6 \cdot \frac{1}{2}\sqrt{x} = 3\sqrt{x}$. Two vertical moves multiply their factors, $6 \cdot \frac{1}{2} = 3$, so the result is one stretch by a factor of 3.

(c) The factors multiply, so the second factor k needs $5k = \frac{1}{2}$, giving $k = \frac{1}{10}$. Since $0 < \frac{1}{10} < 1$, the second move is a vertical compression by a factor of $\frac{1}{10}$. Check: $\frac{1}{10} \cdot 5f(x) = \frac{1}{2}f(x)$.

7. The figure shows the whole graph of $y = f(x)$: line segments with closed dots at the ends (−4, −2) and (4, 0). Let $g(x) = \frac{3}{2}f(x)$.



(a) Sketch the graph of g on the same grid, marking the images of the five marked points.

(b) Give the domain and range of f and of g.

(c) Which points of the graph of f are also on the graph of g? Explain why these and no others.

ANSWER

(a) Segments joining (−4, −3), (−2, 3), (0, 3), (2, −3), (4, 0). (b) Both have domain $-4 \le x \le 4$; f has range $-2 \le y \le 2$ and g has range $-3 \le y \le 3$. (c) The x-intercepts (−3, 0), (1, 0) and (4, 0).

SOLUTION

(a) Each point (x, y) of f moves straight up or down to $\left(x, \frac{3}{2}y\right)$: $(-4, -2) \to (-4, -3)$, $(-2, 2) \to (-2, 3)$, $(0, 2) \to (0, 3)$, $(2, -2) \to (2, -3)$ and (4, 0) stays. A vertical stretch takes a segment to a segment, so join the five new points in order, with closed dots at (−4, −3) and (4, 0).

(b) The inputs do not change, so both domains are $-4 \le x \le 4$. The outputs of f run from −2 to 2; multiplied by $\frac{3}{2}$ they run from −3 to 3. So f has range $-2 \le y \le 2$ and g has range $-3 \le y \le 3$.

(c) A point is on both graphs when $f(x) = \frac{3}{2}f(x)$, that is $\frac{1}{2}f(x) = 0$, so $f(x) = 0$. Those are the x-intercepts of f. The segment from (−4, −2) to (−2, 2) crosses the axis halfway, at (−3, 0); the segment from (0, 2) to (2, −2) crosses at (1, 0); and the end (4, 0) is on the axis. Every other point has $f(x) \neq 0$ and moves to 1.5 times its distance from the axis.

8. Maya draws the profile of a hiking trail for a park map. The true height of the trail above the trailhead is $h(d)$ meters at d meters along the trail (negative means below the trailhead). So that the hills show up, the profile plots $p(d) = 4h(d)$ instead of $h(d)$, with the same horizontal scale. These are heights read off her drawing:

d	0	200	400	600	800
$p(d)$	0	60	140	−20	0

(a) Name the move from the graph of h to the graph of p.

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(b) Make a table of the true heights $h(d)$.

(c) At which listed points does the drawing show the true height? Why there?

(d) Find the average rate of change of p and of h from $d = 200$ to $d = 400$. How do the two compare?

ANSWER

(a) A vertical stretch by a factor of 4. (b) $h(d)$: 0, 15, 35, −5, 0. (c) At $d = 0$ and $d = 800$, where the trail is level with the trailhead. (d) p: 0.4; h: 0.1. The drawing's climb is 4 times as steep.

SOLUTION

(a) Every output of h is multiplied by $4 > 1$ and every input stays, so the graph of p is the graph of h stretched vertically by a factor of 4.

(b) $h(d) = \frac{1}{4}p(d)$, so divide each drawn height by 4:

d	0	200	400	600	800
$h(d)$	0	15	35	−5	0

(c) The drawing is true where $p(d) = h(d)$, that is $4h(d) = h(d)$, so $3h(d) = 0$ and $h(d) = 0$. Of the listed points that is $d = 0$ and $d = 800$, where the trail is at the trailhead's height. A vertical stretch leaves only the points on the horizontal axis in place.

(d) For p: $\frac{140 - 60}{400 - 200} = \frac{80}{200} = 0.4$. For h: $\frac{35 - 15}{200} = \frac{20}{200} = 0.1$. The rate for p is 4 times the rate for h: the rise is multiplied by 4 and the run is not, so the drawing makes the trail look 4 times as steep as it is.

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