Evaluating a logarithm

To evaluate $\log_b x$ is to find the power of b that gives x, as an exact number when x is a neat power of b, or as a decimal when it is not.

WORDS TO KNOW

Evaluate: find the value as a number.

Common logarithm: $\log x$, with base 10. The LOG key on a calculator.

Common base: one smaller number that both the base and the argument are powers of, such as 2 for 8 and 16.

$\log_b x$: the exponent you raise b to in order to get x.

Natural logarithm: $\ln x$, with base $e \approx 2.718$. The LN key.

Consecutive integers: two whole numbers next to each other, such as 3 and 4.

KEY IDEAS

1. Ask which power of the base gives the number

Write the argument as a power of the base, and read off the exponent. The rules for exponents show what a power can look like.

- A reciprocal is a negative power: $\dfrac{1}{b^n} = b^{-n}$.
- A root is a fraction power: $\sqrt[n]{b} = b^{\frac{1}{n}}$.
- $\log_b 1 = 0$ and $\log_b b = 1$ for every allowed base.

The exponent you find is the only one, because the powers of a base always rise or always fall.

2. When both are powers of one smaller base

If the base and the argument are both powers of the same number c, the logarithm is a fraction.

If $b = c^m$ and $x = c^n$, then

$$\log_b x = \frac{n}{m}$$

It works because $b^{\frac{n}{m}} = (c^m)^{\frac{n}{m}} = c^n = x$.

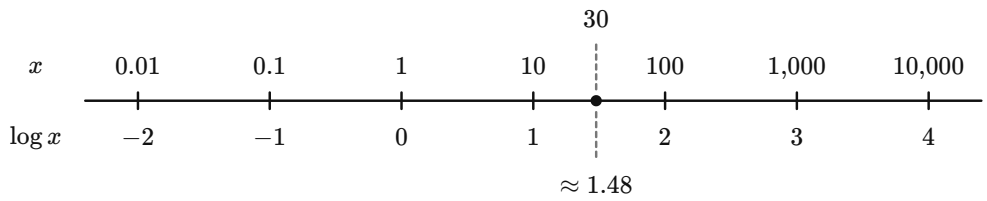
3. Common and natural logs: exact or between two integers

$\log 10^k = k$ and $\ln e^k = k$ for every real k. For other numbers, find the powers on either side. Powers of 10 and of e grow as the exponent grows, so the log sits between those two exponents.

$$10^k \le x < 10^{k+1} \quad \text{gives} \quad k \le \log x < k+1$$

Useful powers of e: $e^1 \approx 2.718$, $e^2 \approx 7.389$, $e^3 \approx 20.09$, $e^4 \approx 54.60$.

A calculator value is rounded, so write it with $\approx$. To check it, raise the base to it: you should get the argument back.



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WATCH OUT

- **Do not flip the question.** $\log_{16} 2$ asks what power of 16 gives 2. That is $\frac{1}{4}$, not 4.
- **A number between 0 and 1 has a negative log** when the base is above 1. With a base below 1 the signs flip: $\log_{1/2} 4 = -2$.
- **The argument must be positive.** $\log(-100)$ and $\ln 0$ have no real value, and a calculator gives an error.
- **Most logs are not exact.** $\log 30$ and $\ln 7$ are irrational, so a decimal for them is only close.

WORKED EXAMPLES

Example 1. Evaluate $\log_3 243$, $\log_2 \frac{1}{64}$ and $\log_3 \left(27\sqrt{3}\right)$ without a calculator.

1. The powers of 3 run 3, 9, 27, 81, 243. So $3^5 = 243$ and $\log_3 243 = 5$.
2. $2^6 = 64$, and a reciprocal is a negative power, so $\frac{1}{64} = 2^{-6}$. So $\log_2 \frac{1}{64} = -6$.
3. Write the root as a power and add exponents: $27\sqrt{3} = 3^3 \cdot 3^{\frac{1}{2}} = 3^{\frac{7}{2}}$. So $\log_3 \left(27\sqrt{3}\right) = \frac{7}{2}$.

Answer: 5, −6 and $\frac{7}{2}$. **Check:** $3^{\frac{7}{2}} = \left(\sqrt{3}\right)^7 = 27\sqrt{3}$.

Example 2. Evaluate $\log_{32} 128$ without a calculator.

1. 128 is not a whole-number power of 32: $32^1 = 32$ and $32^2 = 1{,}024$.
2. Both are powers of 2: $32 = 2^5$ and $128 = 2^7$.
3. By the common-base rule, $\log_{32} 128 = \frac{7}{5}$.

Answer: $\log_{32} 128 = \frac{7}{5}$. **Check:** $32^{\frac{7}{5}} = \left(\sqrt[5]{32}\right)^7 = 2^7 = 128$.

Example 3. Evaluate $\ln \sqrt[3]{e^2}$ exactly. Then say between which two consecutive integers $\log 3{,}700$ and $\ln 9$ lie, and give each to three decimal places.

1. $\sqrt[3]{e^2} = e^{\frac{2}{3}}$, so $\ln \sqrt[3]{e^2} = \frac{2}{3}$.
2. $10^3 = 1000$ and $10^4 = 10{,}000$. Since $1000 < 3{,}700 < 10{,}000$, $\log 3{,}700$ is between 3 and 4. The LOG key gives about 3.568.
3. $e^2 \approx 7.389$ and $e^3 \approx 20.09$. Since 9 is between them, $\ln 9$ is between 2 and 3. The LN key gives about 2.197.

Answer: $\frac{2}{3}$; $\log 3{,}700 \approx 3.568$, between 3 and 4; $\ln 9 \approx 2.197$, between 2 and 3. **Check:** $10^{3.568} \approx 3{,}698$ and $e^{2.197} \approx 9.00$.

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