Algebra 2 Unit 8 worksheet: Evaluating logarithms, common and natural (answer key)

ANSWERS AT A GLANCE

1. (a) 0 (b) −5 (c) −1 (d) 4 (e) −3 (f) −2

2. (a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{1}{6}$ (d) 6 (e) $-\frac{1}{6}$ (f) $-\frac{3}{2}$

3. (a) 0 (b) $-\frac{1}{2}$ (c) $\frac{7}{3}$ (d) $\frac{5}{2}$ (e) 3 (f) 4.5

4. $\log 470$ is between 2 and 3; $\ln 150$ between 5 and 6; $\log_2 30$ between 4 and 5; $\log 0.00093$ between −4 and −3; $\ln 0.2$ between −2 and −1; $\log_5 0.5$ between −1 and 0.
Least to greatest:
$\log 0.00093,\ \ln 0.2,\ \log_5 0.5,\ \log 470,\ \log_2 30,\ \ln 150.$

5. (a) negative (b) positive (c) zero (d) negative (e) positive (f) not a real number (g) not a real number

6. (a) $\log_8 2 = \frac{1}{3}$ (b) $\log 0.001 = -3$ (c) $\ln(-e)$ is not a real number.

7. (a) $\log_2 64 = 6$, $\log_4 64 = 3$, $\log_8 64 = 2$, $\log_{64} 64 = 1$. (b) It would need $b^4 = 64$, but $2^4 = 16$ and $3^4 = 81$, so b would lie strictly between 2 and 3.

WORKED SOLUTIONS

1. Evaluate each logarithm without a calculator.

(a) $\log_{17} 1$ (b) $\log_2 \frac{1}{32}$ (c) $\log_4 0.25$

(d) $\log_{1/5} \frac{1}{625}$ (e) $\log_{0.1} 1000$ (f) $\log_{2/3} \frac{9}{4}$

ANSWER
(a) 0 (b) −5 (c) −1 (d) 4 (e) −3 (f) −2

SOLUTION
In each, ask which power of the base gives the argument.
(a) $17^0 = 1$, so $\log_{17} 1 = 0$.
(b) $2^5 = 32$, and a reciprocal is a negative power: $2^{-5} = \frac{1}{32}$. So $\log_2 \frac{1}{32} = -5$.
(c) $0.25 = \frac{1}{4} = 4^{-1}$, so $\log_4 0.25 = -1$.
(d) $\left(\frac{1}{5}\right)^4 = \frac{1}{625}$, so $\log_{1/5} \frac{1}{625} = 4$. Base and argument are both below 1, and the logarithm is positive.
(e) $0.1 = 10^{-1}$, so $(0.1)^{-3} = 10^3 = 1000$. So $\log_{0.1} 1000 = -3$.
(f) $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$, and the negative exponent flips the fraction: $\left(\frac{2}{3}\right)^{-2} = \frac{9}{4}$. So $\log_{2/3} \frac{9}{4} = -2$.

2. Evaluate each logarithm without a calculator. Write the base and the argument as powers of one smaller base where that helps.

(a) $\log_{16} 8$ (b) $\log_{1/8} \frac{1}{16}$ (c) $\log_4 \sqrt[3]{2}$

(d) $\log_{\sqrt[3]{5}} 25$ (e) $\log_8 \frac{1}{\sqrt{2}}$ (f) $\log_{0.04} 125$

ANSWER
(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{1}{6}$ (d) 6 (e) $-\frac{1}{6}$ (f) $-\frac{3}{2}$

SOLUTION
If $b = c^m$ and $x = c^n$, then $\log_b x = \frac{n}{m}$.
(a) $16 = 2^4$ and $8 = 2^3$, so $\log_{16} 8 = \frac{3}{4}$. Check: $16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$.

Free sample from math.wiki for classroom use. Do not post. math.wiki

(b) $\frac{1}{8} = 2^{-3}$ and $\frac{1}{16} = 2^{-4}$, so $\log_{1/8} \frac{1}{16} = \frac{-4}{-3} = \frac{4}{3}$.

(c) $4 = 2^2$ and $\sqrt[3]{2} = 2^{1/3}$, so $\log_4 \sqrt[3]{2} = \frac{1/3}{2} = \frac{1}{6}$.

(d) $\sqrt[3]{5} = 5^{1/3}$ and $25 = 5^2$, so $\log_{\sqrt[3]{5}} 25 = \frac{2}{1/3} = 6$. Check: $\left(\sqrt[3]{5}\right)^6 = \left(\left(\sqrt[3]{5}\right)^3\right)^2 = 5^2 = 25$.

(e) $8 = 2^3$ and $\frac{1}{\sqrt{2}} = 2^{-1/2}$, so $\log_8 \frac{1}{\sqrt{2}} = \frac{-1/2}{3} = -\frac{1}{6}$.

(f) $0.04 = \frac{1}{25} = 5^{-2}$ and $125 = 5^3$, so $\log_{0.04} 125 = \frac{3}{-2} = -\frac{3}{2}$. Check: $0.04^{-3/2} = 25^{3/2} = 5^3 = 125$.

3. Evaluate exactly. Here log is the common logarithm and ln the natural logarithm.

(a) $\ln 1$

(b) $\log \frac{1}{\sqrt{10}}$

(c) $\log\left(100\sqrt[3]{10}\right)$

(d) $\ln\left(e^2\sqrt{e}\right)$

(e) $\ln \frac{1}{e^{-3}}$

(f) $10^{\log 4.5}$

ANSWER

(a) 0 (b) $-\frac{1}{2}$ (c) $\frac{7}{3}$ (d) $\frac{5}{2}$ (e) 3 (f) 4.5

SOLUTION

Write each argument as one power of 10 (for log) or of e (for ln).

(a) $e^0 = 1$, so $\ln 1 = 0$.

(b) $\frac{1}{\sqrt{10}} = 10^{-1/2}$, so $\log \frac{1}{\sqrt{10}} = -\frac{1}{2}$.

(c) $100\sqrt[3]{10} = 10^2 \cdot 10^{1/3} = 10^{7/3}$, so the value is $\frac{7}{3}$.

(d) $e^2\sqrt{e} = e^2 \cdot e^{1/2} = e^{5/2}$, so the value is $\frac{5}{2}$.

(e) $\frac{1}{e^{-3}} = e^3$, so $\ln \frac{1}{e^{-3}} = 3$.

(f) $\log 4.5$ is the exponent that turns 10 into 4.5, so raising 10 to it gives 4.5 back: $10^{\log 4.5} = 4.5$.

4. Without a calculator, find the two consecutive integers that each of these logarithms lies between. Then list all six from least to greatest. You may use $e^{-2} \approx 0.135$, $e^{-1} \approx 0.368$, $e^5 \approx 148.4$ and $e^6 \approx 403.4$.

$$\log 470 \qquad \ln 150 \qquad \log_2 30 \qquad \log 0.00093 \qquad \ln 0.2 \qquad \log_5 0.5$$

ANSWER

$\log 470$ is between 2 and 3; $\ln 150$ between 5 and 6; $\log_2 30$ between 4 and 5; $\log 0.00093$ between -4 and -3; $\ln 0.2$ between -2 and -1; $\log_5 0.5$ between -1 and 0.

Least to greatest: $\log 0.00093,\ \ln 0.2,\ \log_5 0.5,\ \log 470,\ \log_2 30,\ \ln 150$.

SOLUTION

Each base is above 1, so its powers increase as the exponent increases. Trap each argument between two powers of its base.

$10^2 = 100 < 470 < 1000 = 10^3$, so $\log 470$ is between 2 and 3.

$e^5 \approx 148.4 < 150 < 403.4 \approx e^6$, so $\ln 150$ is between 5 and 6 (only just above 5).

$2^4 = 16 < 30 < 32 = 2^5$, so $\log_2 30$ is between 4 and 5.

$10^{-4} = 0.0001 < 0.00093 < 0.001 = 10^{-3}$, so $\log 0.00093$ is between -4 and -3.

Free sample from math.wiki for classroom use. Do not post. math.wiki

$e^{-2} \approx 0.135 < 0.2 < 0.368 \approx e^{-1}$, so $\ln 0.2$ is between -2 and -1.

$5^{-1} = 0.2 < 0.5 < 1 = 5^0$, so $\log_5 0.5$ is between -1 and 0.

The six intervals do not overlap, so they settle the order: $\log 0.00093 < \ln 0.2 < \log_5 0.5 < \log 470 < \log_2 30 < \ln 150$.

5. Without evaluating, say whether each value is positive, negative, zero, or not a real number. Give a one-line reason.

(a) $\log_7 0.9$ (b) $\log 1.001$ (c) $\log_{2/5} 1$ (d) $\log_{0.5} 6$

(e) $\log_{0.3} 0.2$ (f) $\ln(-4)$ (g) $\log_9 0$

ANSWER

(a) negative (b) positive (c) zero (d) negative (e) positive (f) not a real number (g) not a real number

SOLUTION

For a base above 1, arguments above 1 give positive logarithms and arguments between 0 and 1 give negative ones. For a base between 0 and 1 it is the other way round. Every power of a positive base is positive, so a logarithm of 0 or of a negative number has no real value.

(a) Base 7 > 1 and 0 < 0.9 < 1, so 7 must be raised to a negative power: negative.

(b) Base 10 > 1 and 1.001 > 1: positive (though very small).

(c) $\left(\frac{2}{5}\right)^0 = 1$, so the value is zero.

(d) Base 0.5 < 1, and getting up to 6 > 1 needs a negative power, since $0.5^{-k} = 2^k$: negative.

(e) Base 0.3 < 1 and argument 0.2 < 1: powers of 0.3 fall below 1 only for positive exponents, so positive.

(f) No power of e is negative, so $\ln(-4)$ is not a real number.

(g) No power of 9 is 0, so $\log_9 0$ is not a real number.

6. Each line of Mateo's work has one error. Explain the error and give the correct value.

(a) $\log_8 2 = 3$, because $2^3 = 8$.

(b) $\log 0.001 = -2$, because 0.001 has two zeros after the decimal point.

(c) $\ln(-e) = -1$, because $\ln e = 1$.

ANSWER

(a) $\log_8 2 = \frac{1}{3}$ (b) $\log 0.001 = -3$ (c) $\ln(-e)$ is not a real number.

SOLUTION

(a) Mateo swapped the base and the argument: $2^3 = 8$ says $\log_2 8 = 3$. The question asks which power of 8 gives 2, and $8^{1/3} = \sqrt[3]{8} = 2$, so $\log_8 2 = \frac{1}{3}$.

(b) Counting zeros after the point is not the rule. $0.001 = \frac{1}{1000} = 10^{-3}$, so $\log 0.001 = -3$. (The 1 sits in the third place after the point.)

(c) A minus sign on the argument does not come out in front of the logarithm. Every power of e is positive, so no exponent turns e into $-e$, and $\ln(-e)$ is not a real number.

7. (a) Find every whole-number base $b \geq 2$ for which $\log_b 64$ is a whole number, and give $\log_b 64$ for each.

(b) Explain why no whole-number base b makes $\log_b 64 = 4$.

Free sample from math.wiki for classroom use. Do not post. math.wiki

ANSWER

(a) $\log_2 64 = 6$, $\log_4 64 = 3$, $\log_8 64 = 2$, $\log_{64} 64 = 1$. (b) It would need $b^4 = 64$, but $2^4 = 16$ and $3^4 = 81$, so b would lie strictly between 2 and 3.

SOLUTION

(a) $\log_b 64 = k$ with k a whole number means $b^k = 64 = 2^6$. Since 64 is a power of 2 only, b must be a power of 2, say $b = 2^d$, and then $2^{dk} = 2^6$, so $dk = 6$. The choices are $d = 1, 2, 3, 6$:

$$\log_2 64 = 6, \quad \log_4 64 = 3, \quad \log_8 64 = 2, \quad \log_{64} 64 = 1$$

(Here $k = 0$ is impossible, since $b^0 = 1 \neq 64$.)

(b) $\log_b 64 = 4$ means $b^4 = 64$. Fourth powers of whole numbers increase with the base: $2^4 = 16$ is too small and $3^4 = 81$ is too big, so no whole number works. (The base would be $64^{1/4} = 2\sqrt{2} \approx 2.83$.)

Free sample from math.wiki for classroom use. Do not post. math.wiki