The intermediate value theorem and the bisection method

Most equations cannot be solved by moving terms around. How can you prove a solution exists, and then trap it in a smaller and smaller interval?

WORDS TO KNOW

Intermediate value theorem (IVT): a continuous function on a closed interval takes every height between its two end values.

Root (zero): an input c with $f(c) = 0$.

Sign change: $f(a)$ and $f(b)$ have opposite signs, one negative and one positive.

Bisection: halving an interval again and again, each time keeping the half where the sign still changes.

Midpoint: $m = \frac{a+b}{2}$, the middle of $[a, b]$.

KEY IDEAS

1. The intermediate value theorem

The graph runs without a break from $(a, f(a))$ to $(b, f(b))$. A level line between those heights is below one end and above the other, so the graph must meet it.

> If f is continuous on $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then $f(c) = N$ for at least one c in $[a, b]$.
>
> With $N = 0$: if $f(a)$ and $f(b)$ have opposite signs, f has a zero strictly between a and b.

To use it on an equation $g(x) = k(x)$, look for zeros of $F(x) = g(x) - k(x)$. Always say both things: why F is continuous on the whole closed interval, and the two end values with N between them.

2. Trap the root by halving

> Start with f continuous on $[a, b]$ and a sign change. Find $f(m)$ at the midpoint. If $f(m) = 0$, you are done. If not, keep the half whose ends still have opposite signs. Repeat.
>
> After n steps the interval has length $\frac{b-a}{2^n}$, and its midpoint is within $\frac{b-a}{2^{n+1}}$ of a root.

Bisection is sure but slow: it needs about 3.3 steps for each decimal digit of accuracy, and it finds only one root at a time.

WATCH OUT

- **Continuity on the whole closed interval, ends included.** A jump or an asymptote inside can give a sign change with no zero at all.
- **At least one, not exactly one.** The theorem does not say how many solutions there are or where they are. "Only one" needs a separate reason, such as the function always increasing.
- **A zero without a sign change is invisible.** $(x - 1)^2$ is 0 at $x = 1$ but never negative, so bisection cannot start on it.
- **Keep the half with the sign change,** not the half where f has the same sign at both ends.

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WORKED EXAMPLES

Example 1. Show that $2^x = 3x$ has a solution between 0 and 1. Is it the only solution?

1. Let $F(x) = 2^x - 3x$. An exponential and a polynomial are continuous everywhere, so their difference F is continuous on $[0, 1]$.
2. At the ends: $F(0) = 1 - 0 = 1 > 0$ and $F(1) = 2 - 3 = -1 < 0$.
3. The signs are opposite, so by the IVT, $F(c) = 0$ for some c with $0 < c < 1$. Then $2^c = 3c$.
4. The theorem gives no count. Try $[3, 4]$: $F(3) = 8 - 9 = -1 < 0$ and $F(4) = 16 - 12 = 4 > 0$.

Answer: yes, there is a solution in $(0, 1)$, but it is not the only one: a second solution lies in $(3, 4)$.

Example 2. Show that $f(x) = x^3 - 2x - 5$ has a root in $[2, 3]$. Take four bisection steps, and find how many steps would put the midpoint within 0.01 of the root.

1. f is a polynomial, so it is continuous. $f(2) = 8 - 4 - 5 = -1 < 0$ and $f(3) = 27 - 6 - 5 = 16 > 0$, so a root lies in $(2, 3)$.
2. Halve four times. Here $f(a) < 0 < f(b)$ on every row, and the sign of $f(m)$ decides which end moves to m.

a	b	m	$f(m)$	**keep**
2	3	2.5	5.625	$[2, 2.5]$
2	2.5	2.25	1.8906	$[2, 2.25]$
2	2.25	2.125	0.3457	$[2, 2.125]$
2	2.125	2.0625	−0.3513	$[2.0625, 2.125]$

The final interval has length $\frac{1}{16}$. Its midpoint 2.09375 is within $\frac{1}{32} = 0.03125$ of the root.

For 0.01 you need $\frac{1}{2^{n+1}} \le 0.01$, that is, $2^{n+1} \ge 100$. Since $2^6 = 64$ is too small and $2^7 = 128$ is enough, $n + 1 = 7$ and $n = 6$.

Answer: the root is in $[2.0625, 2.125]$, estimate 2.09375; 6 steps guarantee 0.01. (More steps give about 2.0946.)

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