Continuity at a point

The limit at a point says where the outputs are heading, and the value says where the function actually is. When do the two agree, and how do you check it?

WORDS TO KNOW

Continuous at a: the limit of f at a exists and equals the value $f(a)$.

Discontinuous at a: at least one of the three conditions below fails. The number a is then a **discontinuity** of f.

Rational function: a quotient $\frac{p(x)}{q(x)}$ of two polynomials.

Composition $f\big(g(x)\big)$: the output of the inner function g is fed into the outer function f.

KEY IDEAS

1. One equation, three conditions

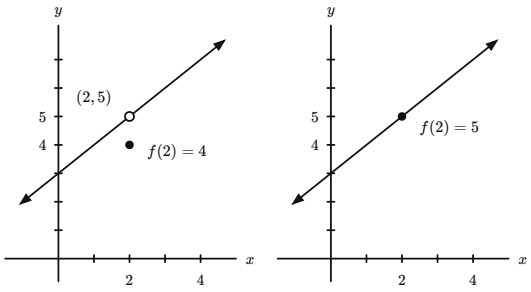
Think of the limit as a prediction made from every value near a except the one at a. Continuity says the prediction comes true.

f is continuous at a when

$$\lim_{x \to a} f(x) = f(a)$$

Condition 1: $f(a)$ is defined. **Condition 2:** $\lim_{x \to a} f(x)$ exists as a number (both one-sided limits exist and agree). **Condition 3:** the limit equals $f(a)$.

Here f is defined on both sides of a. Changing only $f(a)$ can repair a failure of Condition 1 or 3, but never a failure of Condition 2.



2. Building continuous functions

Always continuous: polynomials at every number; $\sin x$, $\cos x$ and b^x ($b > 0$) at every number; $\sqrt[n]{x}$ at every a in its domain except an end of it.

Rational functions $\frac{p(x)}{q(x)}$ are continuous wherever $q(a) \neq 0$, and discontinuous where $q(a) = 0$ (Condition 1 fails).

If f and g are continuous at a, so are $f + g$, $f - g$, kf, fg, and $\frac{f}{g}$ when $g(a) \neq 0$.

Composition. If g is continuous at a and f is continuous at $g(a)$, then $f\big(g(x)\big)$ is continuous at a.

At a point where a function is continuous, you find the limit by plugging in.

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WATCH OUT

- **Find the limit without using $f(a)$.** The limit looks only at inputs near a. Compare it with $f(a)$ only at the end.
- **A limit of ∞ is not a number.** If the outputs grow without bound, Condition 2 fails, and no value of $f(a)$ can fix it.
- **The composition rule needs the outer function continuous at $g(a)$**, not at a.
- **The rules go one way only.** When a piece is not continuous, the rules say nothing: the combination may still be continuous, so check the three conditions directly.

WORKED EXAMPLES

Example 1. Let $f(x) = \dfrac{x^2 - 3x - 28}{x + 4}$ for $x \neq -4$, and $f(-4) = 5$. Is f continuous at −4?

1. Condition 1: $f(-4) = 5$ is defined.
2. Condition 2: substituting −4 gives a top of $16 + 12 - 28 = 0$ and a bottom of 0, the form $\frac{0}{0}$, so factor. For $x \neq -4$,
$$\frac{(x+4)(x-7)}{x+4} = x - 7, \quad \text{so} \quad \lim_{x \to -4} f(x) = -11$$
3. Condition 3: the limit -11 is not the value 5, so this condition fails.

Answer: not continuous at −4: Condition 3 fails. Had the value been $f(-4) = -11$, all three conditions would hold.

Example 2. At which numbers is $q(x) = \dfrac{x + 5}{x^2 + x - 12}$ continuous? Find $\lim_{x \to 6} q(x)$.

1. The top and the bottom are polynomials, so q is continuous wherever the bottom is not 0.
2. Factor the bottom: $x^2 + x - 12 = (x + 4)(x - 3)$, which is 0 only at $x = -4$ and $x = 3$. There q is undefined, so Condition 1 fails.
3. The number 6 is neither, so q is continuous at 6, and the limit is the value: $q(6) = \frac{11}{36+6-12} = \frac{11}{30}$.

Answer: continuous at every number except −4 and 3; $\lim_{x \to 6} q(x) = \frac{11}{30}$.

Example 3. Let $F(x) = \sqrt{\dfrac{3x + 1}{x - 1}}$. Show that F is continuous at 5 and find $F(5)$. Is F continuous at 0?

1. F is a composition: the inner function $g(x) = \frac{3x+1}{x-1}$ feeds the outer function $\sqrt{u}$.
2. At 5 the inner bottom is $4 \neq 0$, so g is continuous at 5, with $g(5) = \frac{16}{4} = 4$. The square root is continuous at 4, since $4 > 0$. By the composition rule, F is continuous at 5.
3. At 0 the inner value is $g(0) = \frac{1}{-1} = -1$, and $\sqrt{-1}$ is not a real number.

Answer: F is continuous at 5 with $F(5) = \sqrt{4} = 2$. $F(0)$ is not defined, so Condition 1 fails and F is not continuous at 0.

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