Continuity of a piecewise function

A function built from pieces can break where one rule hands over to the next. How do you choose its constants so the pieces join?

WORDS TO KNOW

Piece: one rule of a piecewise function, used on its own interval.

Boundary: an input where one rule stops and the next one starts.

Join: at a boundary, both sides arrive at the same height, and the value there is that height.

Corner: a point where a continuous graph changes direction suddenly. A join can have one.

KEY IDEAS

1. Only the boundaries can break

Inside a piece, f agrees with one rule at every nearby input, so f is continuous wherever that rule is. So only the boundaries need work. Each one-sided limit at a boundary c sees only the rule on its own side.

If p is the rule just left of c and q the rule just right of it, continuity at c means

$$\lim_{x \to c^-} p(x) = f(c) = \lim_{x \to c^+} q(x)$$

If a rule is continuous at c, its one-sided limit is its value there. If it reads $\frac{0}{0}$ at c, compute the limit, for example by factoring.

A value given by one of the two rules makes one equation per boundary. A value on a line of its own makes two. Check by putting the constants back: both sides must reach the same height at every boundary.

2. How many answers to expect

How the constant k enters	Number of answers
Linear equations	usually one, but two equations can contradict each other
As k^2 (a quadratic)	two, one or none, by the discriminant
Inside e^{kx}, boundary $c \neq 0$	at most one, since e^t takes each positive value once
Inside $\sin(kx)$, boundary $c \neq 0$	none if the target is outside the sine's range, else infinitely many

If one side runs off to ∞ or $-\infty$ at the boundary, no constant on the other side can join it.

WATCH OUT

- **The constants must not break a piece from the inside.** For example, $\frac{1}{x-k}$ used for $x < 0$ has an asymptote inside its piece when $k < 0$.
- **Use the right rule for $f(c)$.** Read which piece includes the boundary ($\leq$ or $\geq$), or the separate line for $f(c)$.
- **Equal counts of equations and unknowns do not settle it.** Solve, then count the answers.

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WORKED EXAMPLES

Example 1. Let $f(x) = 5 - x^2$ for $x < 1$, $f(x) = ax + b$ for $1 \le x \le 4$, and $f(x) = x^2 - 3$ for $x > 4$. Find a and b so that f is continuous.

1. Each rule is a polynomial, so only the boundaries 1 and 4 need equations.
2. At 1: the left limit is $5 - 1 = 4$. The middle rule includes 1, so $f(1) = a + b$. So $a + b = 4$.
3. At 4: the middle rule includes 4, so $f(4) = 4a + b$. The right limit is $16 - 3 = 13$. So $4a + b = 13$.
4. Subtract the first equation from the second: $3a = 9$, so $a = 3$ and $b = 4 - 3 = 1$.

Answer: $a = 3$ and $b = 1$, so the middle rule is $3x + 1$. **Check:** $3(1) + 1 = 4$ and $3(4) + 1 = 13$.

Example 2. Let $g(x) = k^2 + x$ for $x < 2$ and $g(x) = 3kx - 3$ for $x \ge 2$. Find every k that makes g continuous.

1. For each k both rules are lines, so only $x = 2$ matters.
2. The left limit is $k^2 + 2$. The value and the right limit are $g(2) = 6k - 3$.
3. Set them equal: $k^2 + 2 = 6k - 3$, so $k^2 - 6k + 5 = 0$, which is $(k - 1)(k - 5) = 0$.

Answer: $k = 1$ or $k = 5$. **Check:** with $k = 1$ the rules are $1 + x$ and $3x - 3$, both 3 at 2. With $k = 5$ they are $25 + x$ and $15x - 3$, both 27 at 2.

Example 3. Find c and k so that f is continuous at −2, where $f(x) = \dfrac{x^2 + x - 2}{x + 2}$ for $x < -2$, $f(-2) = c$, and $f(x) = kx + 3$ for $x > -2$.

1. At −2 the top $4 - 2 - 2$ and the bottom are both 0, so compute the left limit. The top factors as $(x + 2)(x - 1)$.
2. For $x \ne -2$, cancel $x + 2$: the left rule is $x - 1$, which tends to −3.
3. The right limit is $-2k + 3$. The value c has its own line, so there are two equations: $-3 = c$ and $-2k + 3 = -3$.

Answer: $c = -3$ and $k = 3$. **Check:** the right rule is $3x + 3$, which gives −3 at −2.

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