# Continuity on an interval

*Big theorems ask for a function continuous on a whole interval, ends included. At an end only one side exists. How do you check continuity there, and on which intervals are the familiar functions continuous?*

## WORDS TO KNOW

**Continuous from the right at $a$:** $\lim_{x \to a^+} f(x) = f(a)$.

**Continuous from the left at $a$:** $\lim_{x \to a^-} f(x) = f(a)$.

**Open interval** $(a, b)$: the numbers strictly between $a$ and $b$, ends left out.

**Closed interval** $[a, b]$: the numbers from $a$ to $b$, both ends included.

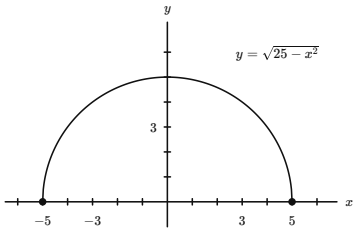
**Continuous function** (no interval named): continuous at every number of its domain, one-sided at any end of the domain.

**Floor** $\lfloor x \rfloor$: the greatest integer less than or equal to $x$.

## KEY IDEAS

### 1. Two-sided inside, one-sided at the ends

At an end of an interval only one side counts. What happens outside the interval is ignored.




$f$ is continuous on $[a, b]$ when it is continuous at every number in $(a, b)$, continuous from the right at $a$, and continuous from the left at $b$.

For $[a, b)$ check only the right side at $a$; for $(a, b]$ only the left side at $b$; for $(a, b)$ no end at all.

$f$ is continuous at $a$ exactly when it is continuous from the left and from the right at $a$.

### 2. The familiar functions are continuous on their domains

| Function | Continuous on |
|---|---|
| polynomials, $\lvert x \rvert$, $\sin x$, $\cos x$, $b^x$ $(b > 0)$ | $(-\infty, \infty)$ |
| rational functions $\frac{p(x)}{q(x)}$ | every interval where $q(x) \neq 0$ |
| $\sqrt[n]{x}$, $n$ even | $[0, \infty)$ |
| $\sqrt[n]{x}$, $n$ odd | $(-\infty, \infty)$ |
| $\ln x$, and $\log_b x$ $(b > 0, b \neq 1)$ | $(0, \infty)$ |

Sums, differences, constant multiples and products of functions continuous on an interval are continuous on it. A quotient is continuous where its bottom is never $0$. A composition is continuous when the inner values stay in an interval where the outer function is continuous.

So a function built from the table's functions is continuous on every interval inside its domain.


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## WATCH OUT

- **One-sided continuity counts only at an end of the interval.** Inside the interval you need two-sided continuity.
- **Defined at the ends is not enough.** At each included end, the one-sided limit must exist and equal the value.
- **Continuous on its domain is not continuous on an interval.** A domain can have a gap, and the function is not continuous on any interval that crosses the gap. Theorems built on continuity always name an interval.
- **Continuous does not mean smooth.** $|x|$ is continuous everywhere, corner and all.

## WORKED EXAMPLES

**Example 1. Find the domain of $f(x) = \sqrt{25 - x^2}$, and show that $f$ is continuous on all of it.**

1. The root needs $25 - x^2 \geq 0$, so the domain is $[-5, 5]$.
2. Inside: for $-5 < a < 5$ the polynomial $25 - x^2$ has a positive value, and the square root is continuous at positive numbers. By the composition result, $f$ is continuous at $a$.
3. At 5: as $x \to 5^-$ the inner values approach 0 and stay in $[0, \infty)$, so $\lim_{x \to 5^-} f(x) = \sqrt{0} = 0 = f(5)$. The same works from the right at −5.

**Answer:** $f$ is continuous on $[-5, 5]$, the half circle in the figure. It has no two-sided limit at 5, but continuity on $[-5, 5]$ never asks for one.

**Example 2. Find the largest intervals on which $F(x) = \sqrt{6 - x} + \dfrac{1}{x + 1}$ is continuous, and check any included end.**

1. The first term needs $6 - x \geq 0$: it is continuous on $(-\infty, 6]$, from the left at 6.
2. The second term is continuous on every interval that avoids −1, where its bottom is 0.
3. By the sum result, the largest intervals are $(-\infty, -1)$ and $(-1, 6]$.
4. At 6 each term is continuous from the left, so the sum is too:

$$\lim_{x \to 6^-} F(x) = \sqrt{0} + \frac{1}{7} = \frac{1}{7} = F(6)$$

**Answer:** $(-\infty, -1)$ and $(-1, 6]$, continuous from the left at 6.

**Example 3. Let $g(x) = x^2 + \lfloor x \rfloor$. Is $g$ continuous on $[-1, 0)$? On $[-1, 0]$?**

1. For $-1 \leq x < 0$, $\lfloor x \rfloor = -1$, so $g(x) = x^2 - 1$, a polynomial. So $g$ is continuous at every number in $(-1, 0)$.
2. At −1 (the left end): $\lim_{x \to -1^+} g(x) = 1 - 1 = 0$, and $g(-1) = 1 + (-1) = 0$. Continuous from the right.
3. At 0 (the right end of $[-1, 0]$): $\lim_{x \to 0^-} g(x) = 0 - 1 = -1$, but $g(0) = 0 + 0 = 0$. Not continuous from the left.

**Answer:** continuous on $[-1, 0)$, which never asks about 0; not continuous on $[-1, 0]$.


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