The formal definition of a limit

Saying the outputs crowd around a height cannot be checked. What exact statement about distances does a limit make, and how do you prove one?

WORDS TO KNOW

ε **(epsilon):** the output tolerance, how close to L the outputs must be.

Distance: $|f(x) - L|$ is how far the output is from L; $|x - a|$ is how far the input is from a.

Scratch work: the side work that finds δ. The proof then shows that this δ works.

δ **(delta):** the input tolerance, how close to a the inputs are kept.

Band and strip: the band is $L - \varepsilon < y < L + \varepsilon$; the strip is $a - \delta < x < a + \delta$.

KEY IDEAS

1. The definition

Let f be defined on an open interval around a, except maybe at a itself.

> $\lim_{x \to a} f(x) = L$ means: for every $\varepsilon > 0$ there is a $\delta > 0$ such that every x with $0 < |x - a| < \delta$ has $|f(x) - L| < \varepsilon$.

In pictures: however thin the band around $y = L$, some strip around $x = a$ is so narrow that the graph inside the strip stays in the band. The point above a is left out, because $0 < |x - a|$ throws out $x = a$.

"For every ε" comes first, so δ may depend on ε. A narrower strip than one that works also works, so you never need the best δ, only one that is small enough.

2. Lines, and the smaller of two numbers

> **Line:** for $m \neq 0$, $|(mx + b) - (ma + b)| = |m|\,|x - a|$, so $\delta = \frac{\varepsilon}{|m|}$ works.
>
> **Other functions:** when the distance is $|x - a|$ times a second factor, agree that $\delta \leq 1$ and find a bound K for the second factor on that strip. Then $\delta = \min\left(1, \frac{\varepsilon}{K}\right)$ works.

3. Showing a number is not the limit

> L is not the limit when one $\varepsilon > 0$ beats every δ: for each $\delta > 0$ there is an x with $0 < |x - a| < \delta$ and $|f(x) - L| \geq \varepsilon$.

WATCH OUT

- **The definition checks a proposed L; it does not find one.** Find L first, from a graph, a table or algebra.
- **Keep the 1 inside the minimum.** The bound K holds only on the strip where $\delta \leq 1$. Without it, a large ε gives a wide strip where K fails.
- **Scratch work is not a proof.** The proof starts from any $\varepsilon > 0$, chooses δ, and shows $|f(x) - L| < \varepsilon$ for every allowed x.
- **For a curve, the strip must be centered.** The graph can leave the band at inputs that sit at different distances from a; the largest δ is the nearer distance.

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WORKED EXAMPLES

Example 1. For $\lim_{x\to 4}\sqrt{x}=2$ and $\varepsilon=0.5$, find the largest δ that works.

1. The band is $1.5<y<2.5$. Since $\sqrt{x}$ is increasing, it is in the band exactly for $1.5^2<x<2.5^2$, that is, $2.25<x<6.25$.
2. The distances from 4 to the ends are $4-2.25=1.75$ and $6.25-4=2.25$.
3. The strip is centered on 4, so it can reach only as far as the nearer end.

Answer: $\delta=1.75$. A larger δ lets in inputs just below 2.25, whose outputs fall below the band.

Example 2. Prove that $\lim_{x\to -3}(4x+5)=-7$. Then find a δ for $\varepsilon=0.6$.

1. Scratch work: $|(4x+5)-(-7)|=|4x+12|=4|x+3|$. So try $\delta=\frac{\varepsilon}{4}$.
2. Proof: let $\varepsilon>0$ and choose $\delta=\frac{\varepsilon}{4}$. If $0<|x+3|<\delta$, then
$$|(4x+5)-(-7)|=4|x+3|<4\delta=\varepsilon$$
3. For $\varepsilon=0.6$ the rule gives $\delta=0.15$.

Answer: the limit is −7, and $\delta=0.15$ works for $\varepsilon=0.6$. **Check:** inputs between −3.15 and −2.85 give outputs between −7.6 and −6.4.

Example 3. Prove that $\lim_{x\to 2}(x^2-x)=2$. Then find the δ your rule gives for $\varepsilon=0.8$.

1. Scratch work: $|x^2-x-2|=|x-2|\,|x+1|$.
2. Agree that $\delta\le 1$. Then $1<x<3$, so $2<x+1<4$ and $|x+1|<4$.
3. Proof: let $\varepsilon>0$ and choose $\delta=\min\left(1,\frac{\varepsilon}{4}\right)$. If $0<|x-2|<\delta$, then $|x+1|<4$ and $|x-2|<\frac{\varepsilon}{4}$, so
$$|x^2-x-2|=|x-2|\,|x+1|<\frac{\varepsilon}{4}\cdot 4=\varepsilon$$
4. For $\varepsilon=0.8$: $\delta=\min(1,0.2)=0.2$.

Answer: the limit is 2, and the rule gives $\delta=0.2$. **Check:** at the strip's ends, $2.2^2-2.2=2.64$ and $1.8^2-1.8=1.44$, both within 0.8 of 2.

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