Growth rates at infinity

Two functions both grow without bound. Which one becomes larger for good, and how do you decide with a limit instead of a few values?

WORDS TO KNOW

Grows faster: f grows faster than g when $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, or, the same thing, $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$. Then g grows slower than f.

Same rate: $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ for a number L with $0 < L < \infty$.

The ladder: logarithms at the bottom, powers in the middle, exponentials at the top.

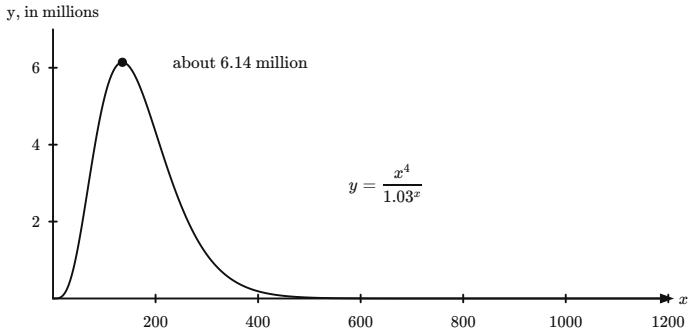
Rung: a step of the ladder. A function on a higher rung grows faster.

KEY IDEAS

1. Compare by the quotient, far out

Here f and g are positive for all large x. The quotient $\frac{f(x)}{g(x)}$ counts how many copies of g fit inside f. Only its long-run behavior counts. On the way it may rise and fall, or stay tiny for a long while. Constant factors do not matter: $1000x^2$ and x^2 grow at the same rate.

The quotient $\frac{x^4}{1.03^x}$ above climbs to about six million near $x = 135$, yet it heads to 0: the exponential grows faster.



2. The ladder

For all constants $p > 0$, $q > 0$ and $a > 1$:

$$\lim_{x\to\infty} \frac{(\ln x)^q}{x^p} = 0 \qquad \lim_{x\to\infty} \frac{x^p}{a^x} = 0$$

Within the powers, x^p grows slower than x^r when $p < r$. Within the exponentials, a^x grows slower than b^x when $1 < a < b$. All logarithms $\log_b x$ with $b > 1$ grow at the same rate.

3. Ranking rules

Positive constant factors do not change a ranking.

If f grows faster than g, and g faster than h, then f grows faster than h.

A sum takes the rank of its fastest term: if g grows slower than f, then $f + g$ grows at the same rate as f.

To find a limit of a quotient of sums, divide the top and the bottom by the fastest term.

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WATCH OUT

- **A few values can point the wrong way.** The slower function can lead for millions of inputs. The limit says who wins, not when.
- **Check the constants.** The ladder needs $p > 0$, $q > 0$ and $a > 1$. For $0 < a < 1$, a^x shrinks toward 0.
- **Some pairs have no ranking.** For $x(2 + \sin x)$ and x, the quotient $2 + \sin x$ keeps swinging between 1 and 3 and has no limit.

WORKED EXAMPLES

Example 1. Find $\lim_{x \to \infty} \dfrac{3 \cdot 5^x + x^7}{10 \cdot 5^x - \sqrt{x}}$.

1. The fastest term on the top and on the bottom is 5^x. Divide every term by it:
$$\frac{3 + \frac{x^7}{5^x}}{10 - \frac{\sqrt{x}}{5^x}}$$
2. By the ladder, $\frac{x^7}{5^x} \to 0$ and $\frac{x^{1/2}}{5^x} \to 0$.
3. The bottom tends to 10, which is not 0, so the quotient law applies.

Answer: the limit is $\frac{3+0}{10-0}$, which is $\frac{3}{10}$.

Example 2. Rank from slowest growing to fastest: x^4, $(\ln x)^4$, 1.5^x, $\sqrt[3]{x}$, $x^3 \ln x$.

1. Check each neighboring pair with a quotient:
$$\frac{(\ln x)^4}{x^{1/3}} \to 0 \qquad \frac{x^3 \ln x}{x^{1/3}} = x^{8/3} \ln x \to \infty \qquad \frac{x^3 \ln x}{x^4} = \frac{\ln x}{x} \to 0 \qquad \frac{x^4}{1.5^x} \to 0$$
2. The first, third and fourth limits are rungs of the ladder ($q = 4$ and $p = \frac{1}{3}$; $q = 1$ and $p = 1$; $p = 4$ and $a = 1.5 > 1$). In the second, both factors grow without bound.
3. So $\sqrt[3]{x}$ is above $(\ln x)^4$, $x^3 \ln x$ is above $\sqrt[3]{x}$, x^4 is above $x^3 \ln x$, and 1.5^x is above x^4.

Answer: $(\ln x)^4$, $\sqrt[3]{x}$, $x^3 \ln x$, x^4, 1.5^x. The transitive rule joins the four steps into one chain.

Example 3. Find $\lim_{x \to 0^+} x^2 \ln x$.

1. As $x \to 0^+$, $x^2 \to 0$ while $\ln x \to -\infty$. The limit laws alone cannot decide.
2. Substitute $x = \frac{1}{u}$. Then $u \to \infty$ exactly when $x \to 0^+$, and $\ln x = -\ln u$.
3. So $x^2 \ln x = -\frac{\ln u}{u^2}$, which tends to 0 by the ladder with $q = 1$ and $p = 2$.

Answer: 0. **Check:** at $x = 0.01$, $0.0001 \cdot \ln 0.01 \approx -0.00046$.

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