Horizontal and oblique asymptotes from limits

What exactly does it mean for a graph to settle toward a line far to the right or left, and how do you find that line with limits instead of a table?

WORDS TO KNOW

Limit at infinity: $\lim_{x\to\infty} f(x) = L$ means the outputs stay as close to L as anyone asks once x is large enough.

Horizontal asymptote: the line $y = L$ when the limit as $x \to \infty$ or as $x \to -\infty$ is L. At most one at each end.

Tolerance t: how close to L the outputs must be. **N:** an input after which they always are that close.

Oblique asymptote: a line $y = mx + b$ with $m \neq 0$ such that $f(x) - (mx + b) \to 0$ at one end.

KEY IDEAS

1. The definition: every tolerance has an N

Draw a band of half-width t around $y = L$. The limit is L when, however thin the band, the graph enters it after some input N and never leaves again. What happens before N does not matter.

$\lim_{x\to\infty} f(x) = L$ means: for every $t > 0$ there is a number N with $|f(x) - L| < t$ for every $x > N$ in the domain.

For $x \to -\infty$, replace $x > N$ by $x < N$.

2. Rewrite until the laws can finish

Basic limit: if $p > 0$, then $\lim_{x\to\infty} \frac{1}{x^p} = 0$. If p is a positive integer, this also holds as $x \to -\infty$.

The limit laws work at $\pm\infty$ when every limit involved is a number.

Forms like $\frac{\infty}{\infty}$ and $\infty - \infty$ say nothing by themselves. So first rewrite:

- **A fraction:** divide every term of the top and the bottom by the highest power of x in the bottom.
- **A square root:** use $\sqrt{x^2} = |x|$, which is x for $x > 0$ and $-x$ for $x < 0$. The two ends can then differ.
- **A difference of two large terms with a root:** multiply the top and the bottom by the conjugate.

3. Two limits find an oblique asymptote

$$m = \lim_{x\to\infty} \frac{f(x)}{x} \qquad b = \lim_{x\to\infty} \big[f(x) - mx\big]$$

If both are numbers and $m \neq 0$, the line $y = mx + b$ is the oblique asymptote as $x \to \infty$. The same works as $x \to -\infty$.

WATCH OUT

- **$\sqrt{x^2}$ is $|x|$, not x.** On the left end, taking x out of a root leaves a minus sign.
- **A table can mislead.** Outputs can drift slowly, as with $\frac{1}{\sqrt{x}}$. Let the algebra decide.
- **A graph may cross its asymptote,** even many times. Only what happens past N counts.

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WORKED EXAMPLES

Example 1. Prove that $\lim_{x\to\infty} \frac{2x+5}{x+1} = 2$ from the definition. Give an N for the tolerance $t = 0.01$, and one for every $t > 0$.

1. Find the distance from 2 over a common bottom:
$$\frac{2x+5}{x+1} - 2 = \frac{2x+5-2(x+1)}{x+1} = \frac{3}{x+1}$$
2. For $x > -1$ the bottom is positive, so the distance $|f(x) - 2|$ is $\frac{3}{x+1}$.
3. This is less than t exactly when $x + 1 > \frac{3}{t}$, that is, when $x > \frac{3}{t} - 1$.
4. For $t = 0.01$: $x > 300 - 1 = 299$.

Answer: $N = 299$ works for $t = 0.01$, and $N = \frac{3}{t} - 1$ works for every $t > 0$. Since $\frac{3}{x+1} > 0$, the graph stays above $y = 2$ on the right.

Example 2. Find the horizontal asymptote at each end of $f(x) = \dfrac{\sqrt{4x^2+3}}{7x-1}$.

1. For $x \neq 0$, factor x^2 out of the root: $\sqrt{4x^2+3} = |x|\sqrt{4 + \frac{3}{x^2}}$. Write the bottom as $x\left(7 - \frac{1}{x}\right)$.
2. As $x \to \infty$, $|x| = x$, and it cancels with the x in the bottom:
$$f(x) = \frac{\sqrt{4 + \frac{3}{x^2}}}{7 - \frac{1}{x}} \to \frac{\sqrt{4}}{7} = \frac{2}{7}$$
3. As $x \to -\infty$, $|x| = -x$, so a minus sign is left, and $f(x) \to -\frac{2}{7}$.

Answer: $y = \frac{2}{7}$ as $x \to \infty$ and $y = -\frac{2}{7}$ as $x \to -\infty$. **Check:** the top is always positive, and the bottom is positive on the far right and negative on the far left.

Example 3. Use two limits to find the oblique asymptote of $f(x) = \dfrac{3x^2 - x + 2}{x - 1}$.

1. Slope: divide the top and the bottom by x^2.
$$m = \lim_{x\to\infty} \frac{3x^2 - x + 2}{x^2 - x} = \frac{3}{1} = 3$$
2. Intercept: put $f(x) - 3x$ over the common bottom.
$$f(x) - 3x = \frac{3x^2 - x + 2 - 3x(x-1)}{x-1} = \frac{2x+2}{x-1} \to 2$$
So $b = 2$.
3. Both limits are the same as $x \to -\infty$, so one line serves at both ends.

Answer: $y = 3x + 2$. **Check:** $(x-1)(3x+2) = 3x^2 - x - 2$, so $f(x) - (3x+2) = \frac{4}{x-1}$, which tends to 0.

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