Kinds of discontinuity: removable, jump, infinite, oscillating

When a function breaks at a point, what kind of break is it, and which breaks can you repair by choosing a better value at that one point?

WORDS TO KNOW

Discontinuity at a: f is defined at every number near a on both sides (at a itself maybe not), but f is not continuous at a.

One-sided limits L^- and L^+: $L^- = \lim_{x \to a^-} f(x)$ and $L^+ = \lim_{x \to a^+} f(x)$.

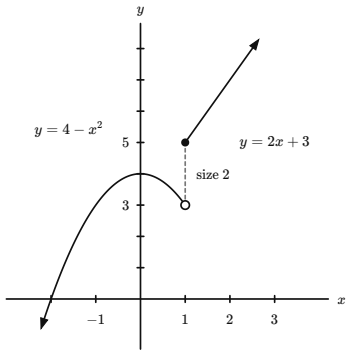
Size of a jump: the gap $|L^+ - L^-|$ between the two one-sided limits.

Continuous extension: the function that keeps every value of f at $x \neq a$ and takes the value $\lim_{x \to a} f(x)$ at a.

KEY IDEAS

1. The two one-sided limits sort every break

Look along the graph toward a from the left and from the right. Each side settles at a height, runs off to $\pm\infty$, or never settles. Every discontinuity is exactly one of four kinds.



Removable: L^- and L^+ are equal numbers, but $f(a)$ is undefined or different.

Jump: L^- and L^+ are both numbers, and $L^- \neq L^+$.

Infinite: at least one one-sided limit is ∞ or $-\infty$.

Oscillating: every other case. A side keeps swinging, as $\sin \frac{1}{x}$ does near 0.

The value $f(a)$ plays no part in the sorting. In a jump it only decides from which side, if either, f is continuous.

2. Only a removable break can be repaired

If $\lim_{x \to a} f(x) = L$, setting the value at a to L makes the function continuous at a, and L is the only value that works.

Changing $f(a)$ never changes a one-sided limit. So a jump, infinite or oscillating break stays, whatever value you choose.

3. Adding or multiplying a continuous function

Let g be continuous at a, and let f jump by J at a. Then $f + g$ jumps by J. If $f(a)$ is defined and $g(a) = 0$, then fg is continuous at a; if $g(a) \neq 0$, fg jumps by $|g(a)|J$.

WATCH OUT

- **A jump needs two numbers.** If one side settles and the other runs off to ∞, the break is infinite, not a jump.
- **The point above a does not decide the kind.** Find both one-sided limits first, then look at $f(a)$.
- **The simplified formula is not the same function.** After canceling, the new formula is the continuous extension. It equals f only where f is defined.

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WORKED EXAMPLES

Example 1. Find and classify every discontinuity of $f(x) = \dfrac{x^2 - x}{x^2 + 2x - 3}$.

1. Factor the bottom: $(x - 1)(x + 3)$. The only candidates are 1 and −3, where f is undefined.
2. Factor the top: $x(x - 1)$. For $x \neq 1, -3$, cancel $x - 1$: $f(x) = \frac{x}{x+3}$.
3. At 1 the simplified form is continuous, so both one-sided limits are $\frac{1}{1+3} = \frac{1}{4}$. The limit exists but $f(1)$ does not: removable.
4. At −3 the top tends to −3 and the bottom $x + 3$ tends to 0 with its own sign. Right of −3 the quotient is negative and huge; left of −3 it is positive and huge.

Answer: removable at 1, with limit $\frac{1}{4}$. Infinite at −3, with $\lim_{x \to -3^+} f(x) = -\infty$ and $\lim_{x \to -3^-} f(x) = \infty$. **Check:** $f(1.01) \approx 0.2519$ and $f(0.99) \approx 0.2481$, both near 0.25.

Example 2. Let $m(x) = x^2 + 3$ for $x \leq -1$ and $m(x) = 2 - x$ for $x > -1$. Classify the break at −1. Is $(x + 1)\, m(x)$ continuous at −1?

1. Each side uses only its own rule: $\lim_{x \to -1^-} m(x) = 1 + 3 = 4$ and $\lim_{x \to -1^+} m(x) = 2 + 1 = 3$.
2. Both are numbers and they differ: a jump of size $|3 - 4| = 1$.
3. The value $m(-1) = 4$ comes from the left rule, so m is continuous from the left, not from the right.
4. The factor $x + 1$ is continuous and is 0 at −1. The one-sided limits of the product are $0 \cdot 4 = 0$ and $0 \cdot 3 = 0$, and its value at −1 is 0.

Answer: a jump of size 1, continuous from the left. The product is continuous at −1.

Example 3. Can a value at −3 make $g(x) = \dfrac{x^2 + x - 6}{|x + 3|}$ continuous there?

1. The top is $(x - 2)(x + 3)$. For $x > -3$, $|x + 3| = x + 3$, so $g(x) = x - 2$, which tends to −5.
2. For $x < -3$, $|x + 3| = -(x + 3)$, so $g(x) = -(x - 2) = 2 - x$, which tends to 5.
3. The one-sided limits are different numbers: a jump of size 10. No value at −3 changes them.

Answer: no. The break is a jump of size 10, and a jump cannot be removed.

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