The squeeze theorem and sin x over x

Some limits wobble, and some mix a sine with a power of x, so the limit laws and factoring both fail. How do you trap such a limit, and why does sin x over x approach 1?

WORDS TO KNOW

Squeeze theorem: a way to find a limit by trapping a function between two simpler ones.

Bounded: stays between two fixed numbers. For example, $-1 \le \cos u \le 1$ for every u.

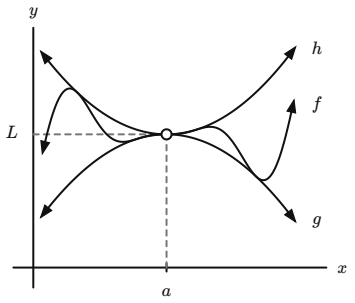
Walls: the two outer functions, one below and one above, that do the trapping.

Radian: the angle measure in which the arc on a unit circle has the same length as the angle. Every limit below uses radians.

KEY IDEAS

1. The squeeze theorem

If the graph of f lies between two walls that both close in on the point (a, L), then f has nowhere else to go. It may wiggle, and it may be undefined at a.



If $g(x) \le f(x) \le h(x)$ for every $x \neq a$ in some open interval around a, and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

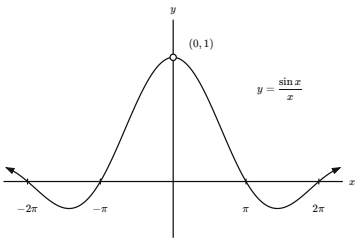
then $\lim_{x \to a} f(x) = L$.

Bounded times zero. If $|u(x)| \le M$ near a and $\lim_{x \to a} v(x) = 0$, then $\lim_{x \to a} u(x)\, v(x) = 0$, even when u has no limit.

The theorem also works for one-sided limits (with the trap on that side) and for $x \to \infty$.

2. Sine over its input

On the unit circle, for $0 < x < \frac{\pi}{2}$, comparing three areas gives $\sin x < x < \tan x$. Dividing and flipping gives the walls $\cos x < \frac{\sin x}{x} < 1$, and both walls tend to 1.



With x in radians:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \qquad \lim_{x \to 0} \frac{\tan x}{x} = 1$$

Rescaling. For a constant $k \neq 0$, $\frac{\sin kx}{kx}$ tends to 1 as $x \to 0$. The input of the sine and the bottom must be the same quantity.

WATCH OUT

- **The walls must meet.** If the walls tend to two different numbers, the squeeze theorem tells you nothing.
- **A bound alone traps nothing.** $\cos \frac{2}{x}$ stays between −1 and 1 but has no limit at 0. The bounded factor needs a partner that goes to 0.
- **Match the input and the bottom.** $\frac{\sin 5x}{x}$ tends to 5, not 1. Write $\frac{\sin 5x}{x} = 5 \cdot \frac{\sin 5x}{5x}$ first.
- **Radians only, and only as the input goes to 0.** A sector of the unit circle has area half its angle only in radians. At any other target, $\frac{\sin x}{x}$ is a quotient whose bottom is not 0, so you substitute.

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WORKED EXAMPLES

Example 1. Find $\lim_{x \to 0} \left(3 + x^2 \sin \dfrac{2}{x} \right)$**.**

1. $\sin \frac{2}{x}$ has no limit at 0, so the sum and product laws cannot be used on it. Squeeze instead.
2. For $x \neq 0$, $-1 \leq \sin \frac{2}{x} \leq 1$. Multiply by $x^2 > 0$, then add 3:
$$3 - x^2 \leq 3 + x^2 \sin \frac{2}{x} \leq 3 + x^2$$
3. Both walls are polynomials with limit 3 at 0.

Answer: 3, by the squeeze theorem. The wobble never stops, but its size x^2 shrinks to nothing.

Example 2. Find $\lim_{x \to 0} \dfrac{\sin 3x}{\tan 5x}$**.**

1. Give each input its own matching bottom, and put back what you borrowed:
$$\frac{\sin 3x}{\tan 5x} = \frac{\sin 3x}{3x} \cdot \frac{5x}{\tan 5x} \cdot \frac{3x}{5x}$$
2. By rescaling, $\frac{\sin 3x}{3x} \to 1$ and $\frac{\tan 5x}{5x} \to 1$, so its reciprocal also tends to 1.
3. For $x \neq 0$ the last factor is $\frac{3}{5}$. The product law gives $1 \cdot 1 \cdot \frac{3}{5}$.

Answer: $\frac{3}{5}$.

Example 3. Find $\lim_{x \to 0} \dfrac{1 - \cos 6x}{x^2}$**.**

1. Substituting gives $\frac{0}{0}$. Multiply the top and the bottom by $1 + \cos 6x$, which is positive near 0, and use $1 - \cos^2 u = \sin^2 u$:
$$\frac{1 - \cos 6x}{x^2} = \frac{\sin^2 6x}{x^2(1 + \cos 6x)} = 36 \left(\frac{\sin 6x}{6x} \right)^2 \cdot \frac{1}{1 + \cos 6x}$$
2. By rescaling and the power law, $\left(\frac{\sin 6x}{6x} \right)^2 \to 1$. By the quotient law, $\frac{1}{1+\cos 6x} \to \frac{1}{2}$.

Answer: $36 \cdot 1 \cdot \frac{1}{2} = 18$. **Check:** at $x = 0.01$ the quotient is about 17.99.

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