Limits by factoring and rationalizing

Substituting the target input into a quotient gives zero over zero, yet the limit is still a definite number. How do you find it, and why is changing the formula allowed?

WORDS TO KNOW

The form $\frac{0}{0}$: the top and the bottom of a quotient both approach 0. It names a situation, not a number.

Indeterminate form: a form like $\frac{0}{0}$ that alone does not decide the limit. The limit may be any number, or may not exist.

Common factor: a factor such as $x - a$ that divides both the top and the bottom.

Conjugate: for $\sqrt{A} - B$, the expression $\sqrt{A} + B$. Their product is $A - B^2$, with no root left.

KEY IDEAS

1. Substitute first, and read what comes out

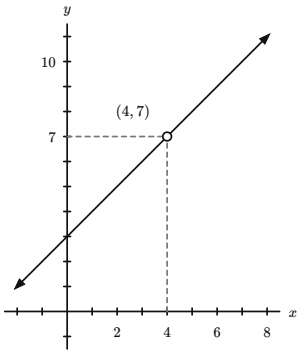
Put $x = a$ into the top and the bottom separately.

Top at a	Bottom at a	What it means
any number	not 0	The limit is the quotient. Done.
not 0	0	No limit as a number. Nothing cancels.
0	0	The form $\frac{0}{0}$. Rewrite, then try again.

If a polynomial is 0 at a, then $x - a$ is a factor of it (the factor theorem). So for a quotient of polynomials, $\frac{0}{0}$ at a means the top and the bottom share the factor $x - a$.

2. Canceling is allowed because the limit skips a

Canceling $x - a$ gives a new function that agrees with the old one at every input other than a itself. The limit never looks at a, so the two limits are the same.



Agreement principle. If $f(x) = g(x)$ for every x near a other than a, and $\lim_{x \to a} g(x)$ exists, then $\lim_{x \to a} f(x)$ equals it.

Steps: factor, cancel $x - a$, substitute again. Still $\frac{0}{0}$? The factor appeared twice; cancel again.

Expand, or combine fractions, first when no factor shows. Useful cube patterns:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \qquad a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

3. When a square root is involved, use the conjugate

Multiply the top and the bottom by the conjugate. The product of the conjugates clears the root and shows the factor $x - a$. Then cancel and substitute.

Keep the other side unexpanded, so the factor $x - a$ stays in sight.

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WATCH OUT

- **Cancel only for $\frac{0}{0}$.** A nonzero top over a bottom near 0 has no common factor, and the limit does not exist as a number.
- **The simplified formula is not the original function.** They differ at a. The limit can be a number while $f(a)$ is still undefined, so never write $f(a)$ equal to the limit.
- **$\frac{0}{0}$ is not 1.** It is not 0 either. It only tells you to rewrite.
- **A zero on top alone is not indeterminate.** If the bottom is not 0, the limit is $0 \div (\text{bottom}) = 0$.

WORKED EXAMPLES

Example 1. Evaluate $\lim_{x \to 3} \dfrac{x^2 + 3x - 18}{x^2 - 10x + 21}$.

1. Substitute: the top is $9 + 9 - 18 = 0$ and the bottom is $9 - 30 + 21 = 0$. The form is $\frac{0}{0}$, so both share $x - 3$.
2. Factor and cancel, for $x \neq 3$:
$$\frac{(x-3)(x+6)}{(x-3)(x-7)} = \frac{x+6}{x-7}$$
3. The new bottom is $3 - 7 = -4 \neq 0$, so substitute: $\frac{9}{-4}$.

Answer: $-\frac{9}{4}$. **Check:** at $x = 2.99$ and $x = 3.01$ the original quotient is about -2.242 and -2.258, on either side of -2.25.

Example 2. Evaluate $\lim_{x \to -2} \dfrac{x^3 + 8}{x^2 - x - 6}$.

1. Substitute: the top is $-8 + 8 = 0$ and the bottom is $4 + 2 - 6 = 0$. Both contain $x + 2$.
2. The top is a sum of cubes with $b = 2$; the bottom is a trinomial:
$$\frac{(x+2)(x^2 - 2x + 4)}{(x+2)(x-3)} = \frac{x^2 - 2x + 4}{x - 3} \quad (x \neq -2)$$
3. Substitute: the top is $4 + 4 + 4 = 12$ and the bottom is -5.

Answer: $-\frac{12}{5}$.

Example 3. Evaluate $\lim_{x \to 1} \dfrac{\sqrt{5x+4} - 3}{x - 1}$.

1. Substitute: the top is $\sqrt{9} - 3 = 0$ and the bottom is 0. The top is not a polynomial, so use its conjugate $\sqrt{5x+4} + 3$, which is at least 3 and never 0.
2. Multiply the top and the bottom by it:
$$\frac{(5x+4) - 9}{(x-1)(\sqrt{5x+4} + 3)} = \frac{5(x-1)}{(x-1)(\sqrt{5x+4} + 3)}$$
3. Cancel $x - 1$: for $x \neq 1$ the quotient is $\frac{5}{\sqrt{5x+4}+3}$. Substitute: $\frac{5}{3+3}$.

Answer: $\frac{5}{6}$. **Check:** at $x = 0.99$ and $x = 1.01$ the original quotient is about 0.8345 and 0.8322, on either side of 0.8333.

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