# Limits from a graph or a table

*A function can refuse an input, or have its value there moved, and its outputs near that input can still crowd around one height. How do you name that height and read it off a graph or a table?*

## WORDS TO KNOW

**Limit:** the height the outputs head toward as the input heads toward a target. $\lim_{x \to a} f(x) = L$ reads "the limit of $f(x)$ as $x$ approaches $a$ is $L$".

**Left-hand limit** $\lim_{x \to a^-} f(x)$: the same, using only inputs less than $a$.

**Right-hand limit** $\lim_{x \to a^+} f(x)$: the same, using only inputs greater than $a$.

**Hole:** a missing point on a graph, drawn as an open circle.

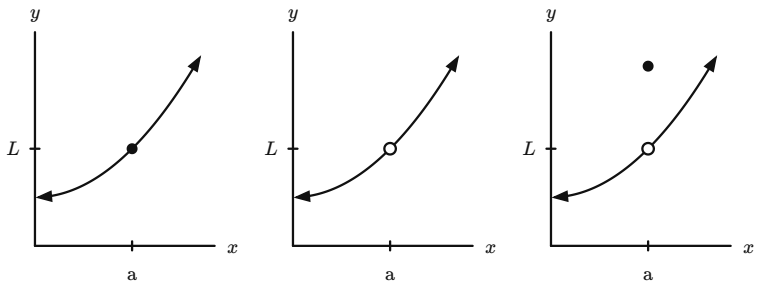
## KEY IDEAS

### 1. A limit ignores the point above $a$

If, for every input $x \neq a$ close enough to $a$, the output $f(x)$ is as close as you like to one number $L$, then $\lim_{x \to a} f(x) = L$. The input never equals $a$, so the value $f(a)$ plays no part.

> **Reading a graph.** Slide one finger along the curve toward $x = a$ from the left, and another from the right. If both head to the same point $(a, L)$, the limit is $L$. Ignore whatever sits directly above $a$.

In all three pictures the limit is $L$. The value $f(a)$ is $L$ in A, undefined in B (a hole), and a different number in C (a moved point).




### 2. Two sides, one limit

> The limit $\lim_{x \to a} f(x)$ exists exactly when both one-sided limits exist and are equal, and then it is their common value.

**A table** of inputs closing in from both sides, such as $a \pm 0.1$, $a \pm 0.01$, $a \pm 0.001$, gives an estimate when both sides settle on the same digits. A table is evidence, not proof. When a factor cancels, the algebra confirms it.


Free sample from math.wiki for classroom use. Do not post. math.wiki

### 3. Three ways a limit does not exist

- **A jump.** Both one-sided limits exist but are different numbers.
- **Unbounded growth.** Near a vertical asymptote the outputs pass every number. $\lim_{x \to a^+} f(x) = \infty$ records how the limit fails. The symbol $\infty$ is not a number, so the limit still does not exist.
- **Endless oscillation.** The outputs keep swinging between heights that stay apart, as $\sin(1/x)$ does near $0$.

**At infinity.** $\lim_{x \to \infty} f(x) = L$ means that far enough to the right every output is as close to $L$ as you like. The curve settles onto the line $y = L$.

**Series.** The sum of a series, when it exists, is the limit of its partial sums: $S = \lim_{n \to \infty} S_n$.

## WATCH OUT

- **The limit is not the value.** The limit at $a$ says nothing about $f(a)$, and $f(a)$ says nothing about the limit.
- **A table can fool you.** Its inputs can all land on special values, such as zeros of sine, while other inputs just as close give very different outputs. Confirm with algebra when you can.
- **"$= \infty$" is not a limit.** It says the outputs grow without bound. Write $\lim_{x \to a} f(x) = \infty$ only when both sides go to $\infty$.

## WORKED EXAMPLES

**Example 1. Let $g(x) = x^2 - 7$ for $x < 4$, $g(4) = 2$, and $g(x) = 2x + 1$ for $x > 4$. Find both one-sided limits at 4, the limit, and $g(4)$.**

1. Just left of 4 the rule is $x^2 - 7$, whose outputs near 4 are near $16 - 7 = 9$. So $\lim_{x \to 4^-} g(x) = 9$.
2. Just right of 4 the rule is $2x + 1$, near $2 \cdot 4 + 1 = 9$. So $\lim_{x \to 4^+} g(x) = 9$.

**Answer:** the sides agree, so $\lim_{x \to 4} g(x) = 9$, while $g(4) = 2$: a moved point. Had the right rule been $x + 1$, the right-hand limit would be 5, and the limit would not exist: a jump.

**Example 2. Estimate $\lim_{x \to -1} \frac{x^2 - x - 2}{x + 1}$ from a table, then confirm it.**

At $x = -1$ the top and the bottom are both $0$, so the function is undefined there. For instance, at $x = -1.1$ the top is $1.21 + 1.1 - 2 = 0.31$, and $0.31 \div (-0.1) = -3.1$.

| $x$ (left) | −1.1 | −1.01 | −1.001 | $x$ (right) | −0.9 | −0.99 | −0.999 |
|---|---|---|---|---|---|---|---|
| output | −3.1 | −3.01 | −3.001 | output | −2.9 | −2.99 | −2.999 |

Both sides settle toward −3. To confirm, factor: $x^2 - x - 2 = (x + 1)(x - 2)$. For $x \neq -1$ the factor $x + 1$ cancels, leaving $x - 2$, whose outputs near −1 are near −3.

**Answer:** the limit is −3, and the function is undefined at −1. The graph is the line $y = x - 2$ with a hole at $(-1, -3)$.

**Example 3. Let $f(x) = \frac{3x + 1}{x + 2}$. Describe $f$ near $x = -2$, and find $\lim_{x \to \infty} f(x)$.**

1. Rewrite: $f(x) = 3 - \frac{5}{x + 2}$. Near $x = -2$ the bottom $x + 2$ is close to $0$.
2. Left of −2, $x + 2$ is tiny and negative, so $-\frac{5}{x + 2}$ is huge and positive: $f(-2.01) = 503$, so $\lim_{x \to -2^-} f(x) = \infty$. Right of −2 it is tiny and positive: $f(-1.99) = -497$, so $\lim_{x \to -2^+} f(x) = -\infty$.
3. The sides go opposite ways, so the limit at −2 does not exist, and not even $\lim_{x \to -2} f(x) = \infty$ can be written.
4. Far right, $f(98) = 3 - \frac{5}{100} = 2.95$ and $f(998) = 2.995$. The gap $\frac{5}{x + 2}$ shrinks toward $0$.

**Answer:** no limit at −2, and $\lim_{x \to \infty} f(x) = 3$.


Free sample from math.wiki for classroom use. Do not post. math.wiki