The limit laws

If you know where two functions are heading, can you say exactly where their sum, product, quotient or composition is heading, without drawing anything?

WORDS TO KNOW

Limit law: a rule that finds the limit of a combination of functions from the limits of the pieces.

Direct substitution: finding a limit by putting a into the function, allowed only when the limit at a equals the value at a.

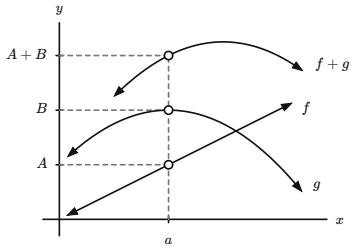
Continuous at a: the limit at a equals the value there, $\lim_{x\to a} f(x) = f(a)$.

Composition $f\big(g(x)\big)$: the output of the inner function g is fed into the outer function f.

KEY IDEAS

1. Do the same arithmetic on the limits

If $f(x)$ is close to A and $g(x)$ is close to B, then $f(x) + g(x)$ is close to $A + B$. The errors can at worst add up, so shrinking them shrinks the error of the sum.



If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, both numbers, and k is a constant:

$$\lim(f+g) = A + B \qquad \lim(f-g) = A - B \qquad \lim kf = kA$$

$$\lim fg = AB \qquad \lim \frac{f}{g} = \frac{A}{B} \text{ only when } B \neq 0$$

Also $\lim_{x\to a} c = c$, $\lim_{x\to a} x = a$, and the power law $\lim \big[f(x)\big]^n = A^n$.

The laws also work for one-sided limits and for $x \to \pm\infty$, as long as every limit involved is a number.

2. Polynomials and other continuous functions: plug in

Direct substitution. $\lim_{x\to a} f(x) = f(a)$ when f is a polynomial, a quotient of polynomials whose bottom is not 0 at a, $\sqrt[n]{x}$ at a in its domain (with $a > 0$ when n is even), $\sin x$, $\cos x$, or b^x with $b > 0$.

3. Roots and compositions

Composition. If $\lim_{x\to a} g(x) = L$ and f is continuous at L, then

$$\lim_{x\to a} f\big(g(x)\big) = f(L)$$

Root law. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{A}$, with $A > 0$ when n is even.

WATCH OUT

- **Both limits must exist.** You cannot split a limit into pieces that have no limit, such as $\frac{1}{x}$ at 0. Infinity is not a number, so no law applies to it.
- **Check the bottom limit first.** If it is 0 and the top limit is not, the outputs grow without bound and the limit does not exist. If both are 0, the form $\frac{0}{0}$ says nothing yet: more algebra is needed.
- **Limit is not value.** Plugging a into the whole expression gives the value. It equals the limit when every piece is continuous at a. If a piece has a moved point at a, the two can differ.
- **The outer function must be continuous at L.** If it jumps at L, the composition rule cannot be used. Look at the actual outputs instead.

Free sample from math.wiki for classroom use. Do not post. math.wiki

WORKED EXAMPLES

Example 1. Find $\lim_{x\to -2}(3x^2+4x-5)$, naming the law behind each step.

1. Each piece has a limit, so the sum and difference laws split it:
$$\lim_{x\to -2} 3x^2 + \lim_{x\to -2} 4x - \lim_{x\to -2} 5$$
2. The constant multiple law, the power law and $\lim_{x\to -2} x = -2$ give
$$3(-2)^2 + 4(-2) - 5 = 12 - 8 - 5$$

Answer: -1, the value of the polynomial at -2, as direct substitution promises.

Example 2. You know only that $\lim_{x\to 6} f(x) = -6$ and $\lim_{x\to 6} g(x) = 2$. Find $\lim_{x\to 6}\big[2f(x) - 5g(x)\big]$, $\lim_{x\to 6}\big[\big(f(x)\big)^2 + x\,g(x)\big]$ and $\lim_{x\to 6}\big[f(x) - g(x)\big]^2$.

1. Constant multiple and difference laws: $2\cdot(-6) - 5\cdot 2 = -12 - 10 = -22$.
2. Power law: $\big(f(x)\big)^2 \to 36$. Product law with $x\to 6$: $x\,g(x)\to 6\cdot 2 = 12$. Sum law: $36 + 12 = 48$.
3. Difference law: $f(x) - g(x) \to -6 - 2 = -8$. Power law: $(-8)^2 = 64$.

Answer: -22, 48 and 64. Nothing was needed about $f(6)$ or $g(6)$.

Example 3. Find $\lim_{x\to 1} \dfrac{\sqrt{x^2+3x+1}}{x+1}$.

1. Top: the inside is a polynomial with $x^2+3x+1 \to 5$, and $5 > 0$, so the root law gives $\sqrt{x^2+3x+1} \to \sqrt{5}$.
2. Bottom: $x + 1 \to 2$.
3. The bottom limit is $2 \neq 0$, so the quotient law applies.

Answer: $\frac{\sqrt{5}}{2}$. **Check:** at $x = 1$ the expression is $\frac{\sqrt{5}}{2}$ too, since every piece is continuous at 1.

Free sample from math.wiki for classroom use. Do not post. math.wiki