When a two-sided limit exists

A function can head toward one height from the left and another from the right. When does it have a limit at all, and how do you find each side from a formula?

WORDS TO KNOW

Left-hand limit $\lim_{x \to a^-} f(x)$: the height $f(x)$ heads toward using only inputs less than a.

Two-sided limit $\lim_{x \to a} f(x)$: the height $f(x)$ heads toward from both sides at once.

Greatest integer function $\lfloor x \rfloor$: the largest integer less than or equal to x. For example $\lfloor 2.7 \rfloor = 2$ and $\lfloor -0.5 \rfloor = -1$.

Right-hand limit $\lim_{x \to a^+} f(x)$: the same, using only inputs greater than a.

Piecewise function: a function with different rules on different parts of its domain.

KEY IDEAS

1. Both sides must agree

Let f be defined near a on both sides (except perhaps at a). Then

$$\lim_{x \to a} f(x) = L \quad \text{exactly when} \quad \lim_{x \to a^-} f(x) = L \ \text{ and } \ \lim_{x \to a^+} f(x) = L$$

If one side has no limit, or the two sides head to different numbers, the limit does not exist. The value $f(a)$ plays no part.

A side that runs off to ∞ or $-\infty$ has no limit as a number, so the two-sided limit fails there too.

2. A one-sided limit sees only the rule on its own side

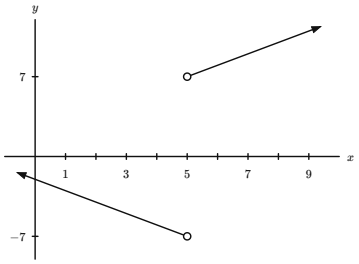
If $f(x)$ follows one rule for every x just left of a, the left-hand limit of f is the left-hand limit of that rule. For a polynomial rule, that is just its value at a, even when the rule's condition (such as $x < a$) leaves a out. The result is the height the side heads toward, not $f(a)$.

To find $\lim_{x \to a} f(x)$ at a boundary of a piecewise function: use the left rule for the left side, the right rule for the right side, and compare.

Away from every boundary, a small window around a uses one rule, so the limit is that rule's value.

3. Absolute values, floors and domain edges

An absolute value is a piecewise rule: $|u| = u$ when $u \geq 0$ and $|u| = -u$ when $u < 0$. Its boundary is where u changes sign. **The floor** $\lfloor x \rfloor$ is constant between integers and steps up by 1 at each integer.



At an integer n: $\lim_{x \to n^-} \lfloor x \rfloor = n - 1$ and $\lim_{x \to n^+} \lfloor x \rfloor = n$.

At an edge of the domain, such as 0 for $\sqrt{x}$, only the one-sided limit from inside is asked: $\lim_{x \to 0^+} \sqrt{x} = 0$.

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WATCH OUT

- **Do not stop after one side.** A left-hand limit alone never settles the two-sided limit. Check the right side too.
- **Do not use $f(a)$.** The value at a, or whether it is defined at all, has no effect on any of the three limits.
- **Plugging a into a rule works only when that rule's limit is its value**, as for a polynomial. A rule like $\frac{1}{x-a}$ at a cannot be handled this way.
- **A few values near a decide nothing.** Locality needs a whole window of inputs on one side that follows one rule.

WORKED EXAMPLES

Example 1. Let $f(x) = x^2 - 2x$ for $x < 3$ and $f(x) = 7 - 2x$ for $x \geq 3$. Find both one-sided limits at 3, the limit, and $f(3)$.

1. Just left of 3 the rule is the polynomial $x^2 - 2x$, so $\lim_{x \to 3^-} f(x) = 9 - 6 = 3$.
2. Just right of 3 the rule is $7 - 2x$, so $\lim_{x \to 3^+} f(x) = 7 - 6 = 1$.
3. The sides are different numbers, so by the criterion the limit does not exist.
4. The condition $x \geq 3$ includes 3, so $f(3) = 7 - 2 \cdot 3 = 1$.

Answer: left 3, right 1, $\lim_{x \to 3} f(x)$ does not exist, and $f(3) = 1$.

Example 2. Let $s(x) = \dfrac{x^2 - x - 6}{|x - 3|}$. Find $\lim_{x \to 3^-} s(x)$, $\lim_{x \to 3^+} s(x)$ and $\lim_{x \to 3} s(x)$.

1. Factor the top: $x^2 - x - 6 = (x - 3)(x + 2)$.
2. For $x < 3$, $x - 3$ is negative, so $|x - 3| = -(x - 3)$. Cancel, since $x - 3 \neq 0$:
$$s(x) = \frac{(x - 3)(x + 2)}{-(x - 3)} = -(x + 2), \quad \text{so} \quad \lim_{x \to 3^-} s(x) = -5$$
3. For $x > 3$, $|x - 3| = x - 3$, so $s(x) = x + 2$ and $\lim_{x \to 3^+} s(x) = 5$.

Answer: left -5, right 5, so the limit does not exist. $s(3)$ is undefined, which would not matter either way.

Example 3. Let $g(x) = x + \lfloor x \rfloor$. Find both one-sided limits at 2, decide whether $\lim_{x \to 2} g(x)$ exists, and find $g(2)$.

1. For $1 \leq x < 2$, $\lfloor x \rfloor = 1$, so $g(x) = x + 1$ and $\lim_{x \to 2^-} g(x) = 3$.
2. For $2 \leq x < 3$, $\lfloor x \rfloor = 2$, so $g(x) = x + 2$ and $\lim_{x \to 2^+} g(x) = 4$.
3. The sides differ, so the limit does not exist.

Answer: left 3, right 4, no limit at 2, and $g(2) = 2 + 2 = 4$. **Check:** $g(1.99) = 2.99$ and $g(2.01) = 4.01$.

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