Infinite limits and vertical asymptotes

When outputs grow past every bound near a point, "the limit does not exist" is true but says too little. What does $\lim_{x\to a} f(x) = \infty$ mean exactly, and how do limits find a vertical asymptote?

WORDS TO KNOW

Infinite limit $\lim_{x\to a} f(x) = \infty$: the outputs can be made larger than any number you name, by taking x close enough to a (with $x \neq a$).

Bound M: the height, as large as anyone likes, that the outputs must pass.

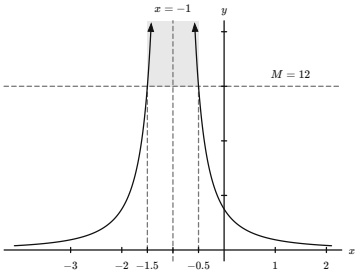
Window δ: a distance from a. Every $x \neq a$ within δ of a must give an output past the bound.

Vertical asymptote: a vertical line $x = a$ that the graph runs along, up or down without end, as x nears a.

KEY IDEAS

1. "$= \infty$" names how a limit fails

∞ is not a number. Writing $\lim_{x\to a} f(x) = \infty$ says the limit does not exist, and says how: the outputs grow without bound and stay large.



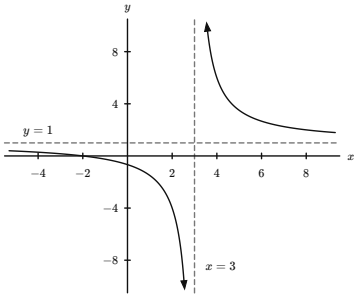
$\lim_{x\to a} f(x) = \infty$ means: for every $M > 0$ there is a $\delta > 0$ with

$$f(x) > M \quad \text{whenever} \quad 0 < |x - a| < \delta$$

For $-\infty$ use $f(x) < -M$. For $x \to a^+$ use $a < x < a + \delta$; for $x \to a^-$ use $a - \delta < x < a$.

The two-sided $\lim_{x\to a} f(x) = \infty$ holds exactly when both one-sided limits are ∞.

2. Nonzero over zero: the sign decides



If the top tends to a number $L \neq 0$ while the bottom tends to 0 and keeps one sign on that side, the quotient tends to ∞ when L and the bottom have the same sign, and to $-\infty$ when the signs are opposite.

For a positive integer n: if n is even, $\frac{1}{(x-a)^n} \to \infty$ from both sides. If n is odd, it tends to $-\infty$ from the left and ∞ from the right.

3. Vertical asymptotes from one-sided limits

The line $x = a$ is a **vertical asymptote** of the graph of f if at least one one-sided limit at a is ∞ or $-\infty$.

One side is enough, and $f(a)$ plays no part.

WATCH OUT

- $\frac{5}{0}$ **is not a value.** It only says the size grows. Check the sign of the bottom on each side before you write ∞ or $-\infty$.
- **Zero over zero is a different question.** If the top also tends to 0, factor first.
- **The limit laws stop at ∞.** The shapes $\infty - \infty$, $0 \cdot \infty$ and $\frac{\infty}{\infty}$ decide nothing.
- **The bottom must keep one sign.** If it changes sign over and over near a, outputs of both signs appear, and the limit is neither ∞ nor $-\infty$.

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WORKED EXAMPLES

Example 1. Prove $\lim_{x\to 2}\dfrac{5}{(x-2)^2}=\infty$. Find a δ for the bound $M=125$, and one for every $M>0$.

1. For $x\neq 2$ the bottom $(x-2)^2$ is positive, so you can multiply by it and divide by M without turning the inequality:
$$\frac{5}{(x-2)^2}>M \iff (x-2)^2<\frac{5}{M} \iff |x-2|<\sqrt{\frac{5}{M}}$$
2. For $M=125$: $\sqrt{5\div 125}=\sqrt{0.04}=0.2$.

Answer: $\delta=0.2$ for $M=125$, and $\delta=\sqrt{\frac{5}{M}}$ in general. So the limit is ∞ and $x=2$ is a vertical asymptote.
Check: $x=2.19$ gives $\frac{5}{0.0361}\approx 138.5>125$.

Example 2. Find every vertical asymptote of $f(x)=\dfrac{x+3}{x^2-4x}$ and the one-sided limits at each.

1. Factor the bottom: $x^2-4x=x(x-4)$, zero at 0 and 4. The top is 3 and 7 there, not 0, so both are nonzero over zero.
2. Near 0: the top tends to 3 and $x-4$ to -4, so f has the sign of $\frac{3}{-4x}$. Left of 0 that is positive, right of 0 negative.
3. Near 4: the top tends to 7 and x to 4, so f has the sign of $\frac{7}{4(x-4)}$. Left of 4 negative, right of 4 positive.

Answer: asymptotes $x=0$ and $x=4$, with
$$\lim_{x\to 0^-}f(x)=\infty,\quad \lim_{x\to 0^+}f(x)=-\infty,\quad \lim_{x\to 4^-}f(x)=-\infty,\quad \lim_{x\to 4^+}f(x)=\infty$$

Example 3. Find $\lim_{x\to\pi^-}\dfrac{\cos x}{\sin x}$ and $\lim_{x\to\pi^+}\dfrac{\cos x}{\sin x}$.

1. As $x\to\pi$, $\cos x\to -1$ and $\sin x\to 0$: nonzero over zero.
2. Just below π the angle is in the second quadrant, where $\sin x>0$: negative over positive.
3. Just above π it is in the third quadrant, where $\sin x<0$: negative over negative.

Answer: $-\infty$ from the left and ∞ from the right. The two-sided limit does not exist, not even as ∞, and $x=\pi$ is a vertical asymptote.

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