Calculus 1 Unit 1 quiz 1: What a limit is: graphs and tables (answer key)

4 questions, 11 points

ANSWERS AT A GLANCE

1. $\lim_{x\to\infty} f(x) = 4$ and $\lim_{x\to-\infty} f(x) = 3$
2. (a) $f(0.1) = 0.79433$, $f(0.01) = 0.95499$, $f(0.001) = 0.99312$, $f(0.0001) = 0.99908$; $\lim_{x\to0^+} f(x) \approx 1$. (b) The domain of f is $x > 0$: there are no inputs to the left of 0, so only the right-hand limit can be asked.
3. (a) $f(x) = -(x+3)$ for $x < 3$, and $f(x) = x + 3$ for $x > 3$. (b) $\lim_{x\to3^-} f(x) = -6$, $\lim_{x\to3^+} f(x) = 6$, and $\lim_{x\to3} f(x)$ does not exist; $f(3)$ is undefined.
4. (a) 1, –1, does not exist, and $f(-2) = 3$. (b) $\lim_{x\to1^-} f(x) = \infty$ and $\lim_{x\to1^+} f(x) = \infty$, so $\lim_{x\to1} f(x) = \infty$; $f(1)$ is undefined. (c) $\lim_{x\to-\infty} f(x) = -3$; as $x \to \infty$, $f(x)$ grows without bound: $\lim_{x\to\infty} f(x) = \infty$.

WORKED SOLUTIONS

1. Let $f(x) = \dfrac{4 \cdot 2^x + 3}{2^x + 1}$. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, and show your reasoning. (2 points)

ANSWER

$\lim_{x\to\infty} f(x) = 4$ and $\lim_{x\to-\infty} f(x) = 3$

SOLUTION

As $x \to -\infty$, 2^x halves with each step to the left, so $\lim_{x\to-\infty} 2^x = 0$. The top approaches 3 and the bottom approaches 1, so $\lim_{x\to-\infty} f(x) = 3$.

As $x \to \infty$, 2^x grows without bound. Divide the top and the bottom by 2^x:

$$f(x) = \frac{4 + 3 \cdot 2^{-x}}{1 + 2^{-x}}$$

Now $2^{-x} \to 0$, so the top approaches 4 and the bottom 1: $\lim_{x\to\infty} f(x) = 4$. A check: $f(10) = \dfrac{4 \cdot 1024 + 3}{1025} \approx 3.9990$.

SCORING

1 point: $\lim_{x\to\infty} f(x) = 4$ with a reason (dividing by 2^x, or a table of large inputs). 1 point: $\lim_{x\to-\infty} f(x) = 3$ with a reason ($2^x \to 0$).

2. Let $f(x) = x^x$ for $x > 0$. Use a calculator. (3 points)

(a) Find $f(x)$ at $x = 0.1$, 0.01, 0.001 and 0.0001, to five decimal places, and estimate $\lim_{x\to0^+} f(x)$.

(b) Explain why the question asks for $\lim_{x\to0^+} f(x)$ and not for $\lim_{x\to0} f(x)$.

ANSWER

(a) $f(0.1) = 0.79433$, $f(0.01) = 0.95499$, $f(0.001) = 0.99312$, $f(0.0001) = 0.99908$; $\lim_{x\to0^+} f(x) \approx 1$. (b) The domain of f is $x > 0$: there are no inputs to the left of 0, so only the right-hand limit can be asked.

SOLUTION

(a) For instance $f(0.1) = 0.1^{0.1} \approx 0.79433$.

x	0.1	0.01	0.001	0.0001
$f(x)$	0.79433	0.95499	0.99312	0.99908

The outputs rise, and their distance from 1 shrinks: about 0.21, 0.045, 0.0069, 0.00092. They are closing in on 1, so $\lim_{x\to0^+} f(x) \approx 1$. (The table is evidence, not proof.)

(b) A two-sided limit needs f defined on both sides near 0. Here f is defined only for $x > 0$, so the inputs can approach 0 only from the right.

SCORING

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1 point: the four values to five decimal places. 1 point: the estimate 1, read from the trend of the outputs. 1 point: (b) no inputs to the left of 0 (domain $x > 0$), so only the one-sided limit makes sense.

3. Let $f(x) = \dfrac{x^2 - 9}{|x - 3|}$. (3 points)

(a) Write $f(x)$ without absolute value bars, once for $x < 3$ and once for $x > 3$.

(b) Find $\lim_{x \to 3^-} f(x)$, $\lim_{x \to 3^+} f(x)$ and $\lim_{x \to 3} f(x)$, and give $f(3)$.

ANSWER

(a) $f(x) = -(x + 3)$ for $x < 3$, and $f(x) = x + 3$ for $x > 3$. (b) $\lim_{x \to 3^-} f(x) = -6$, $\lim_{x \to 3^+} f(x) = 6$, and $\lim_{x \to 3} f(x)$ does not exist; $f(3)$ is undefined.

SOLUTION

(a) The top factors as $(x - 3)(x + 3)$. For $x < 3$, $x - 3$ is negative, so $|x - 3| = -(x - 3)$ and

$$f(x) = \frac{(x - 3)(x + 3)}{-(x - 3)} = -(x + 3).$$

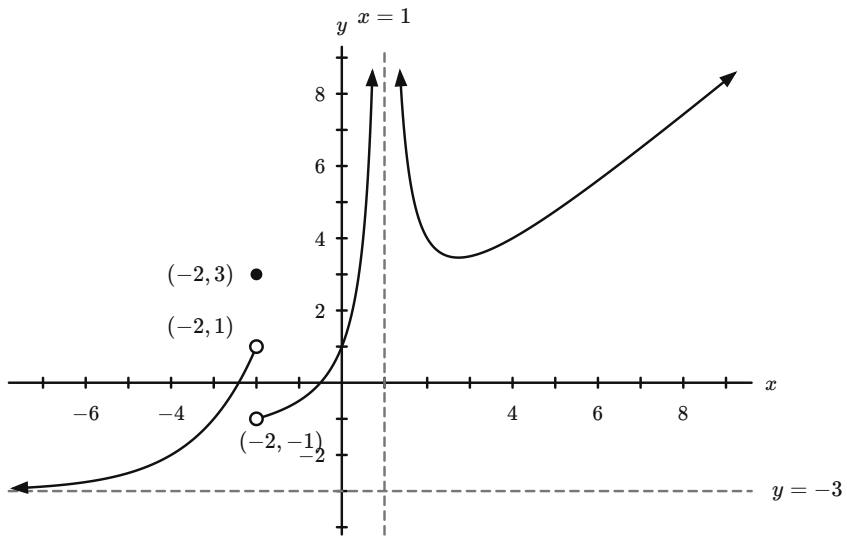
For $x > 3$, $|x - 3| = x - 3$, so $f(x) = x + 3$.

(b) Just left of 3, $f(x) = -(x + 3)$ approaches −6; just right, $f(x) = x + 3$ approaches 6. The one-sided limits are −6 and 6, different numbers, so $\lim_{x \to 3} f(x)$ does not exist. At $x = 3$ the bottom is 0, so $f(3)$ is undefined.

SCORING

1 point: both rewritten rules, with the sign of $x - 3$ on each side. 1 point: −6 and 6 for the one-sided limits. 1 point: the two-sided limit does not exist because they differ, and $f(3)$ undefined.

4. The graph of f is drawn below. The tick marks are 1 unit apart, and the dashed lines are $x = 1$ and $y = -3$. (3 points)



(a) Find $\lim_{x \to -2^-} f(x)$, $\lim_{x \to -2^+} f(x)$, $\lim_{x \to -2} f(x)$ and $f(-2)$.

(b) Describe $f(x)$ as $x \to 1^-$ and as $x \to 1^+$, in limit notation, and give $f(1)$.

(c) Find $\lim_{x \to -\infty} f(x)$, and describe $f(x)$ as $x \to \infty$.

ANSWER

(a) 1, −1, does not exist, and $f(-2) = 3$. (b) $\lim_{x \to 1^-} f(x) = \infty$ and $\lim_{x \to 1^+} f(x) = \infty$, so $\lim_{x \to 1} f(x) = \infty$; $f(1)$ is undefined. (c) $\lim_{x \to -\infty} f(x) = -3$; as $x \to \infty$, $f(x)$ grows without bound: $\lim_{x \to \infty} f(x) = \infty$.

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SOLUTION

(a) From the left the curve rises to the open circle $(-2, 1)$, so $\lim_{x \to -2^-} f(x) = 1$. From the right the next curve starts at the open circle $(-2, -1)$, so $\lim_{x \to -2^+} f(x) = -1$. The sides disagree, so $\lim_{x \to -2} f(x)$ does not exist. The closed dot gives $f(-2) = 3$, a third height that plays no part in the limits.

(b) On both sides of the dashed line $x = 1$ the curve climbs past every height, so $\lim_{x \to 1^-} f(x) = \infty$ and $\lim_{x \to 1^+} f(x) = \infty$; both sides go the same way, so we write $\lim_{x \to 1} f(x) = \infty$, which says how the limit fails to exist. No point sits above 1, so $f(1)$ is undefined.

(c) Far to the left the curve settles onto the dashed line $y = -3$, so $\lim_{x \to -\infty} f(x) = -3$. Far to the right the curve keeps rising with no level line to settle on, so $f(x)$ grows without bound and $\lim_{x \to \infty} f(x) = \infty$: there is no limit.

SCORING

1 point: (a) all four answers right (both one-sided limits, "does not exist", $f(-2) = 3$). 1 point: (b) both one-sided limits written as ∞ in limit notation and $f(1)$ undefined. 1 point: (c) -3, and ∞ (or "grows without bound") as $x \to \infty$.

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