Calculus 1 Unit 1 quiz 2: One sided limits and when a limit exists, The limit laws, Limits by algebra: factoring and rationalizing (answer key)

8 questions, 19 points

ANSWERS AT A GLANCE

1. $\lim_{x \to -2^-} g(x) = 22$ and $\lim_{x \to -2^+} g(x) = 16$; the limit does not exist.
2. $\lim_{x \to 3^-} f(x) = -1$ and $\lim_{x \to 3^+} f(x) = 1$; the limit does not exist.
3. $\frac{8}{9}$
4. $\frac{3}{11}$
5. 6
6. Jin used the value at 2, which the limit never consults. In fact $\lim_{x \to 2^-} f(x) = 2$ and $\lim_{x \to 2^+} f(x) = 2$, so $\lim_{x \to 2} f(x) = 2$, while $f(2) = 7$.
7. $-\frac{9}{13}$
8. $a = 3$ and $b = 4$

WORKED SOLUTIONS

1. Let $g(x) = x^2 - 6\lfloor x \rfloor$. Find $\lim_{x \to -2^-} g(x)$ and $\lim_{x \to -2^+} g(x)$, and decide whether $\lim_{x \to -2} g(x)$ exists. (2 points)

ANSWER

$\lim_{x \to -2^-} g(x) = 22$ and $\lim_{x \to -2^+} g(x) = 16$; the limit does not exist.

SOLUTION

For $-3 \le x < -2$, $\lfloor x \rfloor = -3$, so there $g(x) = x^2 - 6 \cdot (-3)$, a polynomial, and

$$\lim_{x \to -2^-} g(x) = 4 - 6 \cdot (-3) = 22$$

For $-2 \le x < -1$, $\lfloor x \rfloor = -2$, so $g(x) = x^2 - 6 \cdot (-2)$ and

$$\lim_{x \to -2^+} g(x) = 4 - 6 \cdot (-2) = 16$$

The one-sided limits differ (by 6), so by the criterion $\lim_{x \to -2} g(x)$ does not exist.

SCORING

1 point for both one-sided limits, each found from the floor's value on that side; 1 point for "does not exist" justified by the one-sided limits differing.

2. Let $f(x) = \frac{6|x - 3|}{x^2 - 9}$. Find $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$, and decide whether $\lim_{x \to 3} f(x)$ exists. (2 points)

ANSWER

$\lim_{x \to 3^-} f(x) = -1$ and $\lim_{x \to 3^+} f(x) = 1$; the limit does not exist.

SOLUTION

The bottom factors as $(x - 3)(x + 3)$. For $x < 3$, $|x - 3| = -(x - 3)$, so

$$f(x) = \frac{-6(x - 3)}{(x - 3)(x + 3)} = -\frac{6}{x + 3}$$

which heads to $-\frac{6}{6} = -1$. For $x > 3$, $|x - 3| = x - 3$ and $f(x) = \frac{6}{x + 3}$, which heads to 1. The one-sided limits differ, so $\lim_{x \to 3} f(x)$ does not exist.

SCORING

1 point for both one-sided limits, with the absolute value rewritten on each side; 1 point for "does not exist" by the criterion.

Free sample from math.wiki for classroom use. Do not post. math.wiki

3. Find the limit below. Show that the quotient law applies before you use it.

$$\lim_{x \to -3} \frac{x^2 + 6x - 7}{4x - 6}$$

(2 points)

ANSWER

$\frac{8}{9}$

SOLUTION

The top and bottom are polynomials, so their limits are their values at -3:

$$\lim_{x \to -3} (x^2 + 6x - 7) = (-3)^2 + 6 \cdot (-3) - 7 = -16$$
$$\lim_{x \to -3} (4x - 6) = 4 \cdot (-3) - 6 = -18$$

The bottom limit is $-18 \neq 0$, so the quotient law applies:

$$\lim_{x \to -3} \frac{x^2 + 6x - 7}{4x - 6} = \frac{-16}{-18}$$

In lowest terms the limit is $\frac{8}{9}$.

SCORING

1 point for both limits, with the bottom limit -18 shown to be nonzero; 1 point for $\frac{8}{9}$.

4. Evaluate $\lim_{x \to -4} \frac{x^2 + 5x + 4}{3x^2 + 13x + 4}$. (2 points)

ANSWER

$\frac{3}{11}$

SOLUTION

Substituting $x = -4$ makes the numerator and the denominator both 0, so by the factor theorem both contain the factor $x + 4$:

$$\frac{x^2 + 5x + 4}{3x^2 + 13x + 4} = \frac{(x + 4)(x + 1)}{(x + 4)(3x + 1)}.$$

For $x \neq -4$ the quotient equals $\frac{x + 1}{3x + 1}$, whose denominator is $-11 \neq 0$ at -4. So the limit is

$$\frac{-4 + 1}{3 \cdot (-4) + 1} = \frac{-3}{-11} = \frac{3}{11}.$$

SCORING

1 point for factoring both quadratics with the common factor $x + 4$ and canceling it; 1 point for the limit $\frac{3}{11}$.

5. Evaluate $\lim_{x \to -2} \frac{x^2 + 6x + 8}{\sqrt{2x + 13} - 3}$. (2 points)

ANSWER

Free sample from math.wiki for classroom use. Do not post. math.wiki

6

SOLUTION

At $x = -2$ the numerator is 0, and the root is $\sqrt{9} = 3$, so the form is $\dfrac{0}{0}$. The numerator factors as $(x + 2)(x + 4)$. Multiply top and bottom by the conjugate $\sqrt{2x + 13} + 3$, which is at least 3; the bottom becomes $(2x + 13) - 9 = 2(x + 2)$:

$$\frac{(x + 2)(x + 4)\left(\sqrt{2x + 13} + 3\right)}{2(x + 2)}.$$

For $x \neq -2$ this is $\dfrac{(x + 4)\left(\sqrt{2x + 13} + 3\right)}{2}$, and substituting gives $\dfrac{(2)(6)}{2} = 6$.

SCORING

1 point for rationalizing the denominator to $2(x + 2)$ and canceling the factor $x + 2$ from the factored numerator; 1 point for the limit 6.

6. Jin is asked for $\lim_{x \to 2} f(x)$, where

$$f(x) = \begin{cases} \dfrac{6}{x + 1}, & x < 2 \\ 7, & x = 2 \\ \sqrt{x + 2}, & x > 2 \end{cases}$$

He writes: "$f(2) = 7$, so $\lim_{x \to 2} f(x) = 7$." Explain what is wrong with his reasoning, and find the limit correctly, showing both one-sided limits. (3 points)

ANSWER

Jin used the value at 2, which the limit never consults. In fact $\lim_{x \to 2^-} f(x) = 2$ and $\lim_{x \to 2^+} f(x) = 2$, so $\lim_{x \to 2} f(x) = 2$, while $f(2) = 7$.

SOLUTION

A limit at 2 asks about inputs near 2, never about the input 2 itself, so the middle line $f(2) = 7$ tells nothing about the limit.

For $x < 2$ the rule is $\dfrac{6}{x + 1}$, a quotient whose bottom at 2 is $3 \neq 0$, so $\lim_{x \to 2^-} f(x) = \dfrac{6}{3} = 2$.

For $x > 2$ the rule is $\sqrt{x + 2}$, and 2 is inside its domain, so $\lim_{x \to 2^+} f(x) = \sqrt{4} = 2$.

Both one-sided limits exist and equal 2, so by the criterion $\lim_{x \to 2} f(x) = 2$. The middle line gives $f(2) = 7$; the value at 2 has no effect on the limit.

SCORING

1 point: left-hand limit 2 from the quotient rule. 1 point: right-hand limit 2 from the root rule. 1 point: $\lim_{x \to 2} f(x) = 2$ by the criterion, with $f(2) = 7$ stated separately (no point if the limit is given as 7).

7. All that is known about a function f is that $\lim_{x \to -2} f(x) = 5$. Find the limit below, naming the law used at each step.

$$\lim_{x \to -2} \frac{f(x) + x^2}{x\, f(x) - 3}$$

(3 points)

ANSWER

Free sample from math.wiki for classroom use. Do not post. math.wiki

$-\dfrac{9}{13}$

SOLUTION

Top: $\lim_{x\to -2} x = -2$, so the power law gives $x^2 \to (-2)^2$, and the sum law gives

$$f(x) + x^2 \to 5 + (-2)^2 = 9$$

Bottom: the product law gives $x\,f(x) \to (-2)\cdot 5$, and the difference law (with the limit of a constant) gives

$$x\,f(x) - 3 \to (-2)\cdot 5 - 3 = -13$$

The bottom limit is $-13 \neq 0$, so the quotient law gives

$$\lim_{x\to -2} \frac{f(x) + x^2}{x\,f(x) - 3} = \frac{9}{-13}$$

In lowest terms the limit is $-\dfrac{9}{13}$.

SCORING

1 point for the top limit 9 by the sum and power laws; 1 point for the bottom limit -13 by the product and difference laws, with the check that it is not 0; 1 point for $-\dfrac{9}{13}$.

8. Find the constants a and b for which both of these limits hold.

$$\lim_{x\to 2} \frac{ax^2 + b}{x + 2} = 4 \qquad \lim_{x\to -1} \frac{ax^2 + b}{x + 2} = 7$$

(3 points)

ANSWER

$a = 3$ and $b = 4$

SOLUTION

The bottom tends to $2 + 2 = 4$ at 2 and to $-1 + 2 = 1$ at -1. Neither is 0, whatever a and b are, so the quotient law applies at both points. The top tends to $4a + b$ at 2 and to $a + b$ at -1, so the two conditions say

$$\frac{4a + b}{4} = 4, \qquad \frac{a + b}{1} = 7$$

that is, $4a + b = 16$ and $a + b = 7$. Subtracting, $3a = 9$, so $a = 3$ and $b = 4$.

Check: $\lim_{x\to 2} \dfrac{3x^2 + 4}{x + 2} = \dfrac{16}{4} = 4$ and $\lim_{x\to -1} \dfrac{3x^2 + 4}{x + 2} = \dfrac{7}{1} = 7$.

SCORING

1 point for both limits written by the quotient law, with both bottom limits (4 and 1) noted as nonzero; 1 point for the equations $4a + b = 16$ and $a + b = 7$; 1 point for $a = 3$, $b = 4$.

Free sample from math.wiki for classroom use. Do not post. math.wiki