Calculus 1 Unit 1 quiz 3: The squeeze theorem and the limit of sine x over x, Continuity at a point, Continuity on an interval (answer key)

8 questions, 20 points

ANSWERS AT A GLANCE

1. 2
2. (a) $f(3) = 8$ (b) 3
3. f is continuous on $(-\infty, \infty)$; $\lim_{x \to -5} f(x) = -\frac{1}{28}$.
4. $[-5, -1)$, $(-1, 3)$ and $(3, \infty)$; continuous from the right at -5.
5. 1
6. $\frac{3}{2}$
7. Only at $x = 1$. There $\lim_{x \to 1} f\big(g(x)\big) = -2$ but $f\big(g(1)\big) = 0$, so Condition 3 fails. At every other number the composition result makes $f \circ g$ continuous.
8. At 3: $\lim_{x \to 3} f(x) = -\frac{1}{6}$ and $f(3) = -\frac{1}{6}$, so f is continuous, since the limit equals the value. At -3: not continuous, Condition 2 fails, since the limit does not exist.

WORKED SOLUTIONS

1. Find $\lim_{x \to 0} \dfrac{x \tan 8x}{\sin^2 2x}$. (2 points)

ANSWER

2

SOLUTION

Give each trigonometric factor its own matching denominator and put back what was borrowed:

$$\frac{x \tan 8x}{\sin^2 2x} = \frac{\tan 8x}{8x} \cdot \left(\frac{2x}{\sin 2x}\right)^2 \cdot \frac{8x^2}{4x^2}$$

By rescaling the first two factors tend to 1 (the second by the quotient and power laws), and the last is $\frac{8}{4}$ for $x \neq 0$. The limit is 2.

SCORING

1 point for rewriting with matching inputs and denominators; 1 point for the limit 2.

2. Functions f and g are continuous at 3, with $g(3) = 4$ and $f(3) > g(3)$. Also $\lim_{x \to 3} \sqrt{f(x) - g(x)} = 2$. (2 points)

(a) Find $f(3)$.

(b) Find $\lim_{x \to 3} \dfrac{f(x) + g(x)}{f(x) - g(x)}$.

Name the results you use.

ANSWER

(a) $f(3) = 8$ (b) 3

SOLUTION

(a) By the difference result, $f - g$ is continuous at 3, with value $f(3) - 4$, which is positive. Since $\sqrt{u}$ is continuous at every $u > 0$, the composition result makes $\sqrt{f - g}$ continuous at 3, so its limit is its value: $\sqrt{f(3) - 4} = 2$. Squaring, $f(3) - 4 = 4$, so $f(3) = 8$.

(b) By the sum and difference results, the top and the bottom are continuous at 3, with values 8 + 4 = 12 and 8 − 4 = 4 ≠ 0. So the quotient result makes the fraction continuous at 3, and its limit is its value:

$$\frac{12}{4} = 3$$

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SCORING
1 point: (a) $f(3) = 8$, using the composition result to turn the limit into $\sqrt{f(3) - 4}$. 1 point: (b) 3, with the quotient result and the bottom value $4 \neq 0$ checked.

3. Let $f(x) = \dfrac{\sqrt[3]{x+4}}{x^2+3}$. Find the largest interval on which f is continuous, then use continuity to find $\lim_{x \to -5} f(x)$. (2 points)

ANSWER
f is continuous on $(-\infty, \infty)$; $\lim_{x \to -5} f(x) = -\dfrac{1}{28}$.

SOLUTION
The top composes the cube root, continuous on $(-\infty, \infty)$ because 3 is odd, with the polynomial $x + 4$, so it is continuous on $(-\infty, \infty)$; negative values under an odd root cause no trouble. The bottom is a polynomial with $x^2 + 3 \geq 3 > 0$, never 0. By the quotient result, f is continuous on $(-\infty, \infty)$.

So f is continuous at -5, and the limit is the value:

$$\lim_{x \to -5} f(x) = f(-5) = \frac{\sqrt[3]{-5+4}}{(-5)^2+3} = \frac{\sqrt[3]{-1}}{28} = -\frac{1}{28}$$

SCORING
1 point for $(-\infty, \infty)$ with both reasons (an odd root is continuous everywhere, the bottom is never 0); 1 point for the limit $-\dfrac{1}{28}$ found as the value $f(-5)$.

4. Let $f(x) = \dfrac{\sqrt{x+5}}{x^2 - 2x - 3}$. Find the largest intervals on which f is continuous, and say which one-sided condition holds at any end an interval includes. (2 points)

ANSWER
$[-5, -1)$, $(-1, 3)$ and $(3, \infty)$; continuous from the right at -5.

SOLUTION
The top needs $x + 5 \geq 0$, so $x \geq -5$; as the square root of a polynomial it is continuous on $[-5, \infty)$, from the right at -5.

The bottom factors as $(x + 1)(x - 3)$, which is 0 at -1 and 3, both to the right of -5. By the quotient result, f is continuous on every interval inside $[-5, \infty)$ that avoids -1 and 3:

$$[-5, -1), \quad (-1, 3), \quad (3, \infty)$$

At -5, f is continuous from the right, with $f(-5) = 0$.

SCORING
1 point for the domain $x \geq -5$ and the zeros -1 and 3 of the bottom; 1 point for the three intervals with -5 included and continuity from the right named there.

5. The greatest integer function $\lfloor t \rfloor$ gives the greatest integer less than or equal to t. Use the squeeze theorem to find $\lim\limits_{x \to 0^+} x \left\lfloor \dfrac{1}{x} \right\rfloor$. (3 points)

ANSWER
1

SOLUTION

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Every number t satisfies $t - 1 < \lfloor t \rfloor \le t$. With $t = \dfrac{1}{x}$:

$$\frac{1}{x} - 1 < \left\lfloor \frac{1}{x} \right\rfloor \le \frac{1}{x}$$

For $x > 0$, multiply through by the positive number x, which keeps both inequalities:

$$1 - x < x \left\lfloor \frac{1}{x} \right\rfloor \le 1$$

As $x \to 0^+$, $1 - x \to 1$ and the constant wall is 1. By the squeeze theorem for a one-sided limit, $\lim_{x \to 0^+} x \left\lfloor \dfrac{1}{x} \right\rfloor = 1$.

SCORING

1 point for the bounds $\dfrac{1}{x} - 1 < \lfloor \dfrac{1}{x} \rfloor \le \dfrac{1}{x}$; 1 point for multiplying by $x > 0$ to get walls $1 - x$ and 1, with the reason the inequalities keep their direction; 1 point for both wall limits and the conclusion 1 by the squeeze theorem.

6. A function g satisfies $\cos x \le g(x) \le 1 + x^2$ for every x. Find $\lim\limits_{x \to 0} \dfrac{g(x) \sin 3x}{2x}$. (3 points)

ANSWER

$\dfrac{3}{2}$

SOLUTION

First $\lim_{x \to 0} g(x)$. The lesson showed $\cos x \to 1$, and $1 + x^2 \to 1$, so the walls meet and the squeeze theorem gives $\lim_{x \to 0} g(x) = 1$.

Next, $\dfrac{\sin 3x}{2x} = \dfrac{3}{2} \cdot \dfrac{\sin 3x}{3x} \to \dfrac{3}{2} \cdot 1 = \dfrac{3}{2}$ by rescaling.

By the product law,

$$\lim_{x \to 0} \frac{g(x) \sin 3x}{2x} = 1 \cdot \frac{3}{2} = \frac{3}{2}$$

SCORING

1 point for $\lim_{x \to 0} g(x) = 1$ by the squeeze theorem with both wall limits; 1 point for $\dfrac{\sin 3x}{2x} \to \dfrac{3}{2}$ by rescaling; 1 point for the product law and the answer $\dfrac{3}{2}$.

7. Let $g(x) = x^2 - 2x$, and let $f(u) = \dfrac{u^2 - 1}{u + 1}$ for $u \ne -1$, with $f(-1) = 0$. Find every number x at which $f\big(g(x)\big)$ is not continuous, and name the condition that fails there. Justify your answer. (3 points)

ANSWER

Only at $x = 1$. There $\lim_{x \to 1} f\big(g(x)\big) = -2$ but $f\big(g(1)\big) = 0$, so Condition 3 fails. At every other number the composition result makes $f \circ g$ continuous.

SOLUTION

The inner function g is a polynomial, so it is continuous at every number. Near any $u \ne -1$, f is given by a rational formula whose bottom $u + 1$ is not 0, so f is continuous at every $u \ne -1$. By the composition result, $f \circ g$ is continuous at every x with $g(x) \ne -1$.

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Solve $g(x) = -1$: $x^2 - 2x + 1 = (x-1)^2 = 0$, so $x = 1$ is the only such number. There the composition result does not apply, because f is not continuous at -1: $\lim_{u \to -1} f(u) = \lim_{u \to -1}(u - 1) = -2 \neq 0 = f(-1)$. Check the conditions directly at $x = 1$.

Condition 1: $f\big(g(1)\big) = f(-1) = 0$. For $x \neq 1$, $g(x) \neq -1$, so $f\big(g(x)\big) = g(x) - 1 = x^2 - 2x - 1$, a polynomial. So $\lim_{x \to 1} f\big(g(x)\big) = 1 - 2 - 1 = -2$, and Condition 2 holds. The limit -2 is not the value 0, so Condition 3 fails and $f \circ g$ is not continuous at 1.

SCORING

1 point: the composition result gives continuity wherever $g(x) \neq -1$, and solving $(x-1)^2 = 0$ leaves only $x = 1$. 1 point: at 1, finds the limit -2 (using $g(x) \neq -1$ for $x \neq 1$, or the limit of f at -1). 1 point: at 1, compares with the value $f(-1) = 0$ and concludes not continuous (Condition 3).

8. Let $f(x) = \dfrac{x^2 - 7x + 12}{x^2 - 9}$ for $x \neq \pm 3$, with $f(3) = -\dfrac{1}{6}$ and $f(-3) = 0$. Decide whether f is continuous at 3 and whether it is continuous at -3. Where it is not, name the condition that fails. (3 points)

ANSWER

At 3: $\lim_{x \to 3} f(x) = -\dfrac{1}{6}$ and $f(3) = -\dfrac{1}{6}$, so f is continuous, since the limit equals the value. At -3: not continuous, Condition 2 fails, since the limit does not exist.

SOLUTION

Factor: $x^2 - 7x + 12 = (x-3)(x-4)$ and $x^2 - 9 = (x-3)(x+3)$. For $x \neq \pm 3$ the factor $x - 3$ cancels, and $f(x) = \dfrac{x-4}{x+3}$.

At 3: Condition 1 holds, $f(3) = -\dfrac{1}{6}$. The limit looks only at $x \neq 3$, where the canceled form holds, and that form is continuous at 3, so $\lim_{x \to 3} f(x) = \dfrac{3-4}{6} = -\dfrac{1}{6}$. Condition 2 holds, and the limit equals the value, so Condition 3 holds: f is continuous at this point.

At -3: Condition 1 holds, $f(-3) = 0$. Near -3 the top $x - 4$ heads to $-7 \neq 0$ while the bottom $x + 3$ heads to 0, negative on the left and positive on the right. So the outputs grow without bound, with opposite signs on the two sides, and the limit does not exist: Condition 2 fails. A value at -3 cannot help, since there is no limit for it to match.

SCORING

1 point: factors and cancels, and finds $\lim_{x \to 3} f(x) = -\dfrac{1}{6}$. 1 point: at 3, compares the limit with the value $f(3) = -\dfrac{1}{6}$ and concludes correctly (continuous when they are equal, Condition 3 fails when they are not). 1 point: at -3, shows the top does not head to 0 while the bottom does, so the limit does not exist (Condition 2), even though $f(-3)$ is defined.

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