Calculus 1 Unit 1 quiz 4: Infinite limits and vertical asymptotes, Discontinuities and how to remove them, Making a piecewise function continuous (answer key)

8 questions, 19 points

ANSWERS AT A GLANCE

1. $\lim_{x\to 3^-} g(x) = -\infty$ and $\lim_{x\to 3^+} g(x) = \infty$
2. (a) $\lim_{x\to 0} k(x) = -\infty$ (b) $-\infty$ from the left and ∞ from the right
3. Removable at -4 and at 4; the limit at each is 4. The extension is $F(x) = \dfrac{|x|+4}{2}$ for every real x.
4. $k = -5$; the pieces meet at height 4.
5. The top also tends to 0, so the result on nonzero over zero does not apply. For $x \neq 2$ the function is $\dfrac{x+2}{x-2}$, so $\lim_{x\to 2^-} = -\infty$, $\lim_{x\to 2^+} = \infty$, and $\lim_{x\to 2}$ does not exist.
6. (a) 5 and 2, so f has a jump of size 3 at 3. (b) $c = 9$ only. (c) 15.
7. $a = 3$ and $b = 3$
8. Infinite at 0: $\lim_{x\to 0^-} f(x) = \infty$, $\lim_{x\to 0^+} f(x) = -\infty$. Infinite at 3: $\lim_{x\to 3^-} f(x) = -\infty$, $\lim_{x\to 3^+} f(x) = \infty$. The bottom has the factor $x-3$ twice and the top only once, so a factor $x-3$ is left in the bottom after canceling, and the function still runs off to infinity at 3.

WORKED SOLUTIONS

1. Let $g(x) = \dfrac{x^2-6}{x^2+2x-15}$. Find $\lim_{x\to 3^-} g(x)$ and $\lim_{x\to 3^+} g(x)$, and explain each sign. (2 points)

ANSWER

$\lim_{x\to 3^-} g(x) = -\infty$ and $\lim_{x\to 3^+} g(x) = \infty$

SOLUTION

Factor the bottom: $x^2+2x-15 = (x-3)(x+5)$, which is zero at $x = 3$. As $x \to 3$ the top tends to $9 - 6 = 3 \neq 0$ and $x+5$ tends to $8 > 0$, while $x-3$ tends to 0: it is negative for $x < 3$ and positive for $x > 3$. Nonzero over zero, so the size grows without bound.

From the left the bottom is negative, so $g(x)$ has the sign opposite to 3: $\lim_{x\to 3^-} g(x) = -\infty$. From the right the bottom is positive, so $g(x)$ has the sign of 3: $\lim_{x\to 3^+} g(x) = \infty$.

SCORING

1 point for factoring the bottom as $(x-3)(x+5)$ and $\lim_{x\to 3^-} g(x) = -\infty$ with the signs of the top and of each factor shown; 1 point for $\lim_{x\to 3^+} g(x) = \infty$ with the sign of $x-3$ on that side. A bare answer with no sign work earns at most 1 point.

2. Let $k(x) = \dfrac{e^x+1}{|x|\,(x-3)}$. (2 points)

(a) Find $\lim_{x\to 0} k(x)$, or explain why it does not exist.

(b) Find $\lim_{x\to 3^-} k(x)$ and $\lim_{x\to 3^+} k(x)$.

ANSWER

(a) $\lim_{x\to 0} k(x) = -\infty$ (b) $-\infty$ from the left and ∞ from the right

SOLUTION

(a) As $x \to 0$ the top tends to $e^0 + 1 = 2$, and $x-3$ tends to -3, while $|x|$ tends to 0 and is positive on both sides. So the bottom tends to 0 through negative values on both sides: positive over negative from both sides, and $\lim_{x\to 0} k(x) = -\infty$. The absolute value acts like an even power: no sign change at 0.

(b) As $x \to 3$ the top tends to $e^3 + 1 > 0$ and $|x| \to 3$, while $x - 3 \to 0$, negative on the left and positive on the right. So $\lim_{x\to 3^-} k(x) = -\infty$ and $\lim_{x\to 3^+} k(x) = \infty$.

SCORING

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1 point for $-\infty$ at 0 from both sides, with $|x| > 0$ and $x - 3 \to -3$ shown (an answer of "does not exist because the sides differ" earns nothing); 1 point for $-\infty$ and ∞ at 3 with the sign of $x - 3$ on each side.

3. Let $f(x) = \dfrac{x^2 - 16}{2|x| - 8}$. Find and classify every discontinuity of f. Then write the continuous extension of f as one formula that holds for every real x. (2 points)

ANSWER

Removable at -4 and at 4; the limit at each is 4. The extension is $F(x) = \dfrac{|x| + 4}{2}$ for every real x.

SOLUTION

The bottom is $2\big(|x| - 4\big)$, which is 0 at $x = \pm 4$, so these are the candidates; f is continuous everywhere else. Since $x^2 = |x|^2$, the top is $|x|^2 - 16 = \big(|x| - 4\big)\big(|x| + 4\big)$. So for $x \neq \pm 4$,

$$f(x) = \frac{|x| + 4}{2}$$

This form is continuous everywhere, so at both 4 and -4 the limit is $\dfrac{4 + 4}{2} = 4$, while f is undefined: both discontinuities are removable. The continuous extension takes the value 4 at both points, which is exactly what $\dfrac{|x| + 4}{2}$ gives there, so $F(x) = \dfrac{|x| + 4}{2}$ for every real x.

4. Find the constant k that makes f continuous, and the height at which the pieces meet.

$$f(x) = \begin{cases} kx + 19, & x < 3 \\ x^2 + k, & x \geq 3 \end{cases}$$

(2 points)

ANSWER

$k = -5$; the pieces meet at height 4.

SOLUTION

Both rules are polynomials, so only $x = 3$ matters. The left limit is $3k + 19$ and $f(3) = 9 + k$. Continuity says

$$3k + 19 = 9 + k, \qquad 2k = -10, \qquad k = -5$$

Check: $f(3) = 9 - 5 = 4$, and the left rule gives $(-5) \cdot 3 + 19 = 4$.

5. Samir writes: "$\lim\limits_{x \to 2} \dfrac{x^2 - 4}{(x - 2)^2} = \infty$, because the bottom $(x - 2)^2$ tends to 0 and is positive on both sides of 2." Explain what is wrong with his reasoning, and find the correct one-sided limits at 2 and the two-sided limit. (3 points)

ANSWER

The top also tends to 0, so the result on nonzero over zero does not apply. For $x \neq 2$ the function is $\dfrac{x + 2}{x - 2}$, so $\lim_{x \to 2^-} = -\infty$, $\lim_{x \to 2^+} = \infty$, and $\lim_{x \to 2}$ does not exist.

SOLUTION

The result Samir is using needs the top to tend to a nonzero number. Here $x^2 - 4 \to 0$, so the shape is $\dfrac{0}{0}$, which decides nothing until the common factor is removed.

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Factor: $\dfrac{x^2-4}{(x-2)^2} = \dfrac{(x-2)(x+2)}{(x-2)^2} = \dfrac{x+2}{x-2}$ for $x \neq 2$. Now the top tends to $4 \neq 0$ and the bottom $x-2$ changes sign at 2. Positive over negative on the left: $\lim_{x\to 2^-} = -\infty$. Positive over positive on the right: $\lim_{x\to 2^+} = \infty$.

The sides go opposite ways, so $\lim_{x\to 2} \dfrac{x^2-4}{(x-2)^2}$ does not exist, not even as ∞. The line $x = 2$ is still a vertical asymptote.

SCORING

1 point for saying the top also tends to 0, so the nonzero over zero result does not apply; 1 point for canceling to $\dfrac{x+2}{x-2}$; 1 point for $-\infty$ from the left, ∞ from the right, and "does not exist" for the two-sided limit.

6. Let $f(x) = 2x - 1$ for $x < 3$ and $f(x) = x^2 - 7$ for $x \geq 3$. (3 points)

(a) Find $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$.

(b) Find every constant c for which $(x^2 - c)f(x)$ is continuous at 3.

(c) Find the size of the jump of $(x + 2)f(x)$ at 3.

ANSWER

(a) 5 and 2, so f has a jump of size 3 at 3. (b) $c = 9$ only. (c) 15.

SOLUTION

(a) $\lim_{x\to 3^-} f(x) = 2 \cdot 3 - 1 = 5$ and $\lim_{x\to 3^+} f(x) = 9 - 7 = 2$, with $f(3) = 2$. Both are numbers and differ: a jump of size 3.

(b) The factor $g(x) = x^2 - c$ is continuous with $g(3) = 9 - c$. The product has one-sided limits $5(9 - c)$ and $2(9 - c)$, which are equal only when $9 - c = 0$. For $c = 9$ both limits are 0 and the value is $f(3) \cdot 0 = 0$, so the product is continuous. For any other c it has a jump of size $3|9 - c| \neq 0$. So $c = 9$.

(c) The factor $x + 2$ is continuous with value 5 at 3, so the one-sided limits are $5 \cdot 5 = 25$ and $5 \cdot 2 = 10$: a jump of size $15 = 5 \cdot 3$.

SCORING

1 point: (a) the one-sided limits 5 and 2. 1 point: (b) $c = 9$, with a reason that rules out every other c (the one-sided limits $5(9 - c)$ and $2(9 - c)$, or the product result). 1 point: (c) 15.

7. Find a and b so that f is continuous on its whole domain $[-3, \infty)$.

$$f(x) = \begin{cases} a\sqrt{x+3}, & -3 \leq x < 1 \\ bx^2 + a, & 1 \leq x \leq 2 \\ 21 - 3x, & x > 2 \end{cases}$$

(3 points)

ANSWER

$a = 3$ and $b = 3$

SOLUTION

Inside the pieces nothing breaks: $a\sqrt{x+3}$ is continuous on (−3, 1) and continuous from the right at the end −3, and the other rules are polynomials. Only the boundaries 1 and 2 need equations.

At 1. The middle rule includes 1: $f(1) = b + a$. The left limit is $a\sqrt{4} = 2a$. So $2a = a + b$, that is $b = a$.

At 2. The middle rule includes 2: $f(2) = 4b + a$. The right limit is 21 − 6 = 15. So $4b + a = 15$.

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With $b = a$: $5a = 15$, so $a = 3$ and $b = 3$. Check: at 1 both sides arrive at 6 ($3 \cdot 2$ and $3 + 3$); at 2 both arrive at 15 ($12 + 3$ and $21 - 6$).

SCORING

1 point: the equation at 1, $2a = a + b$ (left limit from the root, value from the middle rule). 1 point: the equation at 2, $4b + a = 15$. 1 point: solves to $a = 3$, $b = 3$.

8. Let $f(x) = \dfrac{x^2 - 9}{x^3 - 6x^2 + 9x}$. Find and classify every discontinuity of f, giving the one-sided limits at each. Explain why the discontinuity at 3 is not removable, even though a factor $x - 3$ cancels. (2 points)

ANSWER

Infinite at 0: $\lim_{x \to 0^-} f(x) = \infty$, $\lim_{x \to 0^+} f(x) = -\infty$. Infinite at 3: $\lim_{x \to 3^-} f(x) = -\infty$, $\lim_{x \to 3^+} f(x) = \infty$. The bottom has the factor $x - 3$ twice and the top only once, so a factor $x - 3$ is left in the bottom after canceling, and the function still runs off to infinity at 3.

SOLUTION

The bottom is $x(x^2 - 6x + 9) = x(x - 3)^2$ and the top is $(x - 3)(x + 3)$, so the candidates are 0 and 3. For $x \neq 0, 3$,

$$f(x) = \frac{x + 3}{x(x - 3)}$$

Near 0 this behaves like $\dfrac{3}{x \cdot (-3)} = -\dfrac{1}{x}$: ∞ from the left, $-\infty$ from the right. Infinite.

Near 3 it behaves like $\dfrac{6}{3(x - 3)} = \dfrac{2}{x - 3}$: $-\infty$ from the left, ∞ from the right. Infinite. Canceling removed only one of the two factors $x - 3$ in the bottom; the simplified form still has bottom 0 at 3 and top $6 \neq 0$, so the one-sided limits are infinite and no value at 3 can remove the break.

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