Calculus 1 Unit 1 quiz 5: Limits at infinity and horizontal asymptotes, Comparing growth rates at infinity (answer key)

8 questions, 22 points

ANSWERS AT A GLANCE

1. $\dfrac{9}{2}$
2. $c = 48$
3. $y = \dfrac{1}{3}$ as $x \to \infty$ and $y = 1$ as $x \to -\infty$.
4. (b) $N = 6\sqrt{11}$ (about 19.9) (c) $N = \sqrt{\frac{20}{t} - 4}$
5. $c = 2$
6. $\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0^-} f(x) = 0$; $\lim_{x \to 0} f(x)$ does not exist.
7. (a) $\dfrac{12}{5}$ (b) Slower: $\lim_{x \to \infty} \dfrac{f(x)}{h(x)} = 0$.
8. $\ln(x^{10})$, $x^{0.2}$, $\ln(2^x)$, $\ln(x^x)$

WORKED SOLUTIONS

1. Find the limit below.

$$\lim_{x \to -\infty} \frac{(3x - 5)(6x + 2)}{(2x + 8)^2}$$

(2 points)

ANSWER

$\dfrac{9}{2}$

SOLUTION

Divide the top and the bottom by x^2, giving one factor x to each of the two factors on top and putting x^2 inside the square on the bottom:

$$\frac{\left(3 - \frac{5}{x}\right)\left(6 + \frac{2}{x}\right)}{\left(2 + \frac{8}{x}\right)^2}.$$

The basic limit with the positive integer power 1 holds as $x \to -\infty$, so each $\dfrac{\text{constant}}{x} \to 0$. By the product law the top tends to $3 \cdot 6 = 18$, by the power law the bottom tends to $2^2 = 4 \neq 0$, and the quotient law gives $\dfrac{9}{2}$.

SCORING

1 point for dividing by x^2 correctly so each small term has a power of x below it; 1 point for $\dfrac{9}{2}$.

2. Find the constant c for which

$$\lim_{x \to \infty} \frac{c \cdot 4^x + x^6}{2^{2x+4} - x \ln x} = 3.$$

(2 points)

ANSWER

$c = 48$

SOLUTION

First put the bottom on the same base: $2^{2x+4} = 2^4 \cdot \left(2^2\right)^x = 16 \cdot 4^x$. So 4^x is the fastest term on top and on the bottom; divide by it:

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$$\frac{c + \dfrac{x^6}{4^x}}{16 - \dfrac{x \ln x}{4^x}}.$$

By the ladder $\dfrac{x^6}{4^x} \to 0$, and $\dfrac{x \ln x}{4^x} = \dfrac{\ln x}{x} \cdot \dfrac{x^2}{4^x} \to 0 \cdot 0 = 0$. So the limit is $\dfrac{c}{16}$. Setting $\dfrac{c}{16} = 3$ gives $c = 48$.

SCORING

1 point for rewriting 2^{2x+4} as $16 \cdot 4^x$ and showing the limit is $\dfrac{c}{16}$ (each leftover piece goes to 0 by the ladder); 1 point for $c = 48$.

3. Find the horizontal asymptote at each end of the graph of $f(x) = \dfrac{\sqrt{x^2 + 5}}{\sqrt{4x^2 - x} + x}$. (3 points)

ANSWER

$y = \dfrac{1}{3}$ as $x \to \infty$ and $y = 1$ as $x \to -\infty$.

SOLUTION

For $x \neq 0$, factor x^2 out under each root and use $\sqrt{x^2} = |x|$:

$$\sqrt{x^2 + 5} = |x|\sqrt{1 + \tfrac{5}{x^2}}, \qquad \sqrt{4x^2 - x} = |x|\sqrt{4 - \tfrac{1}{x}}.$$

As $x \to \infty$, $|x| = x$. Divide top and bottom by x:

$$f(x) = \frac{\sqrt{1 + \frac{5}{x^2}}}{\sqrt{4 - \frac{1}{x}} + 1} \to \frac{1}{2 + 1} = \frac{1}{3}.$$

As $x \to -\infty$, $|x| = -x$, so the top is $-x\sqrt{1 + \frac{5}{x^2}}$ and the bottom is $-x\sqrt{4 - \frac{1}{x}} + x$. Divide top and bottom by $-x$:

$$f(x) = \frac{\sqrt{1 + \frac{5}{x^2}}}{\sqrt{4 - \frac{1}{x}} - 1} \to \frac{1}{2 - 1} = 1.$$

The root law applies at each step because the expressions under the roots tend to 1 and 4, both positive. The asymptotes are $y = \dfrac{1}{3}$ on the right and $y = 1$ on the left.

SCORING

1 point for $y = \dfrac{1}{3}$ as $x \to \infty$ with the division shown; 1 point for using $\sqrt{x^2} = |x| = -x$ for negative x; 1 point for $y = 1$ as $x \to -\infty$.

4. Let $f(x) = \dfrac{5x^2}{x^2 + 4}$, so that $\lim_{x \to \infty} f(x) = 5$. (3 points)

(a) Show that $|f(x) - 5| = \dfrac{20}{x^2 + 4}$ for every x.

(b) In the definition of this limit, find the smallest N that works for the tolerance $t = 0.05$. Give it exactly.

(c) For $0 < t < 5$, find an N in terms of t that works.

ANSWER

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(b) $N = 6\sqrt{11}$ (about 19.9) (c) $N = \sqrt{\frac{20}{t} - 4}$

SOLUTION

(a) $f(x) - 5 = \dfrac{5x^2 - 5(x^2+4)}{x^2+4} = \dfrac{-20}{x^2+4}$, and $x^2 + 4 > 0$, so $|f(x) - 5| = \dfrac{20}{x^2+4}$.

(b) $\dfrac{20}{x^2+4} < 0.05$ exactly when $x^2 + 4 > 400$, that is, $x^2 > 396$. For positive x this means $x > \sqrt{396} = 6\sqrt{11}$. So every $x > 6\sqrt{11}$ is within 0.05 of 5. Any smaller N fails, because it lets in inputs x with $N < x \le 6\sqrt{11}$, where $x^2 \le 396$ (or $x \le 0$) and the distance is at least 0.05. The smallest N is $6\sqrt{11} \approx 19.9$.

(c) $\dfrac{20}{x^2+4} < t$ exactly when $x^2 > \dfrac{20}{t} - 4$, which is positive because $t < 5$. So $N = \sqrt{\frac{20}{t} - 4}$ works: every $x > N$ is positive with $x^2 > \dfrac{20}{t} - 4$. (Check: $t = 0.05$ gives $\sqrt{396}$, as in (b).)

SCORING

1 point for the distance $\dfrac{20}{x^2+4}$ with the sign handled; 1 point for $N = 6\sqrt{11}$; 1 point for $N = \sqrt{\frac{20}{t} - 4}$.

5. Find the constant c for which

$$\lim_{x \to \infty} \left(\sqrt{x^2 + cx} - \sqrt{x^2 - 4x + 1} \right) = 3.$$

(3 points)

ANSWER

$c = 2$

SOLUTION

Both roots grow without bound, so rewrite with the conjugate before using any law:

$$\sqrt{x^2 + cx} - \sqrt{x^2 - 4x + 1} = \frac{(x^2 + cx) - (x^2 - 4x + 1)}{\sqrt{x^2 + cx} + \sqrt{x^2 - 4x + 1}} = \frac{(c+4)x - 1}{\sqrt{x^2 + cx} + \sqrt{x^2 - 4x + 1}}.$$

For large positive x each root is x times a root that tends to 1: $\sqrt{x^2 + cx} = x\sqrt{1 + \frac{c}{x}}$ and $\sqrt{x^2 - 4x + 1} = x\sqrt{1 - \frac{4}{x} + \frac{1}{x^2}}$. Divide top and bottom by x:

$$\frac{(c+4) - \frac{1}{x}}{\sqrt{1 + \frac{c}{x}} + \sqrt{1 - \frac{4}{x} + \frac{1}{x^2}}} \to \frac{c+4}{2}.$$

Setting $\dfrac{c+4}{2} = 3$ gives $c = 2$. Check: with $c = 2$ the limit is $\dfrac{6}{2} = 3$.

SCORING

1 point for the conjugate rewrite with top $(c+4)x - 1$; 1 point for the limit $\dfrac{c+4}{2}$; 1 point for $c = 2$.

6. Let $f(x) = x^3 e^{1/x}$ for $x \neq 0$. Find $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$, and decide whether $\lim_{x \to 0} f(x)$ exists.
(3 points)

ANSWER

$\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0^-} f(x) = 0$; $\lim_{x \to 0} f(x)$ does not exist.

SOLUTION

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From the right, $x^3 \to 0$ while $e^{1/x} \to \infty$: the shape $0 \cdot \infty$. Put $x = \dfrac{1}{u}$, so $u \to \infty$ exactly when $x \to 0^+$:

$$x^3 e^{1/x} = \frac{e^u}{u^3}.$$

By the ladder $\dfrac{u^3}{e^u} \to 0$, and its positive reciprocal grows without bound, so $\lim_{x \to 0^+} f(x) = \infty$.

From the left, $\dfrac{1}{x} \to -\infty$, so $e^{1/x} \to 0$, and $x^3 \to 0$. This is not an indeterminate shape: by the product law $\lim_{x \to 0^-} f(x) = 0 \cdot 0 = 0$.

The one-sided limits differ (one is not even a number), so $\lim_{x \to 0} f(x)$ does not exist.

SCORING

1 point for the right-hand limit ∞ by a substitution that gives an exponential over a power; 1 point for the left-hand limit 0 by the product law; 1 point for concluding the two-sided limit does not exist because the one-sided limits differ.

7. Functions f, g and h are positive for all large x, and all that is known about them is

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{h(x)}{g(x)} = 4.$$

(a) Find $\displaystyle\lim_{x \to \infty} \frac{f(x) + 3h(x)}{g(x) + h(x)}$.

(b) Does f grow faster than h, slower, or at the same rate? Justify with a limit. (3 points)

ANSWER

(a) $\dfrac{12}{5}$ (b) Slower: $\lim_{x \to \infty} \dfrac{f(x)}{h(x)} = 0$.

SOLUTION

(a) Divide top and bottom by $g(x)$:

$$\frac{f(x) + 3h(x)}{g(x) + h(x)} = \frac{\dfrac{f(x)}{g(x)} + 3 \cdot \dfrac{h(x)}{g(x)}}{1 + \dfrac{h(x)}{g(x)}}.$$

The top goes to $0 + 3 \cdot 4 = 12$ and the bottom to $1 + 4 = 5 \neq 0$, so the limit is $\dfrac{12}{5}$.

(b) $\dfrac{f(x)}{h(x)} = \dfrac{f(x)}{g(x)} \cdot \dfrac{g(x)}{h(x)}$. The first factor goes to 0, and the second is the reciprocal of a quotient going to 4, so it goes to $\dfrac{1}{4}$. The product goes to $0 \cdot \dfrac{1}{4} = 0$: f grows slower than h.

SCORING

1 point for dividing by $g(x)$ to write the quotient in terms of the two given limits; 1 point for $\dfrac{12}{5}$; 1 point for writing $\dfrac{f}{h}$ as $\dfrac{f}{g} \cdot \dfrac{g}{h}$ with limit 0 and the verdict slower.

8. Rank these four functions from slowest growing to fastest. Rewrite each logarithm first, then justify each neighboring pair in your ranking with the limit of a quotient.

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$$\ln(x^x), \quad x^{0.2}, \quad \ln(2^x), \quad \ln(x^{10})$$

(3 points)

ANSWER

$\ln(x^{10})$, $x^{0.2}$, $\ln(2^x)$, $\ln(x^x)$

SOLUTION

First rewrite the logarithms (for $x > 0$): $\ln(x^{10}) = 10 \ln x$, $\ln(2^x) = x \ln 2$ and $\ln(x^x) = x \ln x$. Taking a logarithm moves a power down to a logarithm and an exponential down to a power.

- $\dfrac{\ln(x^{10})}{x^{0.2}} = 10 \cdot \dfrac{\ln x}{x^{0.2}} \to 10 \cdot 0 = 0$ by the ladder with $q = 1$ and $p = 0.2$, so $x^{0.2}$ is above $\ln(x^{10})$. The exponent 10 inside the logarithm is only a constant factor.
- $\dfrac{x^{0.2}}{\ln(2^x)} = \dfrac{x^{0.2}}{x \ln 2} = \dfrac{1}{\ln 2 \cdot x^{0.8}} \to 0$, so $\ln(2^x)$ is above $x^{0.2}$.
- $\dfrac{\ln(x^x)}{\ln(2^x)} = \dfrac{x \ln x}{x \ln 2} = \dfrac{\ln x}{\ln 2} \to \infty$, so $\ln(x^x)$ is above $\ln(2^x)$.

By the transitive rule the chain holds: $\ln(x^{10})$, $x^{0.2}$, $\ln(2^x)$, $\ln(x^x)$.

SCORING

1 point for each of the three neighboring pairs placed in the right order and justified by a quotient with the correct limit (∞ or 0). A pair placed by guess or by plugging in one value earns no point.

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