Calculus 1 Unit 1 quiz 6: The intermediate value theorem, The epsilon delta definition of a limit (answer key)

7 questions, 20 points

ANSWERS AT A GLANCE

1. Let $g(x) = f(x) - x^2$, continuous on $[1, 4]$. $g(1) = 1 > 0 > -8 = g(4)$, so by the intermediate value theorem $g(c) = 0$, that is $f(c) = c^2$, for some c with $1 < c < 4$.
2. $\delta = \sqrt{2} - 1$ (about 0.414)
3. Let $F(x) = \sqrt{x+4} - x + 5$, continuous on $[\,8,\ 9\,]$. $F(8) = \sqrt{12} - 3 > 0$ and $F(9) = \sqrt{13} - 4 < 0$, so by the intermediate value theorem $F(c) = 0$, that is $\sqrt{c+4} = c - 5$, for some c with $8 < c < 9$.
4. (a) $\delta = \dfrac{5\varepsilon}{9}$ works (proof in the key). (b) Within 0.5 degrees Celsius of 25.
5. (a) $|2x^2 - x - 3| = |x + 1|\,|2x - 3|$ (b) $K = 7$ (c) $\delta = \min\left(1, \dfrac{\varepsilon}{7}\right)$; for $\varepsilon = 0.14$, $\delta = 0.02$
6. (a) f is continuous on $[2, 5]$ (a quotient of polynomials with $x - 1 \neq 0$ there) and $\dfrac{15}{2} < 8 < 9$, so such a c exists. (b) No: the theorem promises a solution when the target is between the end values and says nothing when it is not. In fact $f(x) = 7$ has two solutions in $[2, 5]$, $x = 3$ and $x = 4$.
7. (a) The third row: after $p(1.25) < 0$ she should keep $[1.25, 1.5]$, but she kept $[1, 1.25]$, where p is negative at both ends. (b) Row 3 is $a = 1.25$, $b = 1.5$, $m = 1.375$, $p(1.375) \approx 0.3809$; the final interval is $[1.25, 1.375]$ with midpoint 1.3125.

WORKED SOLUTIONS

1. A function f is continuous on $[1, 4]$, with $f(1) = 2$ and $f(4) = 8$. Show that $f(c) = c^2$ for some c strictly between 1 and 4. (2 points)

ANSWER

Let $g(x) = f(x) - x^2$, continuous on $[1, 4]$. $g(1) = 1 > 0 > -8 = g(4)$, so by the intermediate value theorem $g(c) = 0$, that is $f(c) = c^2$, for some c with $1 < c < 4$.

SOLUTION

The equation $f(x) = x^2$ asks for a zero of $g(x) = f(x) - x^2$. Since f and x^2 are continuous on $[1, 4]$, so is their difference g. At the ends,

$$g(1) = f(1) - 1 = 2 - 1 = 1 > 0, \qquad g(4) = f(4) - 16 = 8 - 16 = -8 < 0.$$

By the intermediate value theorem there is c in $[1, 4]$ with $g(c) = 0$, and c is not an end since $g(1) \neq 0$ and $g(4) \neq 0$. So $1 < c < 4$ and $f(c) = c^2$.

SCORING

1 point: $g(x) = f(x) - x^2$ set up and named continuous on $[1, 4]$ as a difference of continuous functions. 1 point: $g(1) = 1 > 0$ and $g(4) = -8 < 0$, and the conclusion $f(c) = c^2$ with $1 < c < 4$.

2. The limit $\lim_{x \to 3}(x^2 - 4x) = -3$ holds. For $\varepsilon = 1$, find the largest δ that works. Give it exactly. (2 points)

ANSWER

$\delta = \sqrt{2} - 1$ (about 0.414)

SOLUTION

The band is $-4 < y < -2$. Write $x^2 - 4x = (x - 2)^2 - 4$. The bottom condition $(x - 2)^2 - 4 > -4$ holds for every $x \neq 2$: the graph touches the bottom of the band only at its lowest point, $x = 2$. The top condition $(x - 2)^2 - 4 < -2$ means $(x - 2)^2 < 2$, that is, $2 - \sqrt{2} < x < 2 + \sqrt{2}$. So the inputs near 3 whose outputs lie in the band are $2 < x < 2 + \sqrt{2}$.

The strip must be centered on 3. The distances to the ends are 3 − 2 = 1 on the left and $2 + \sqrt{2} - 3 = \sqrt{2} - 1 \approx 0.414$ on the right. The nearer end decides: $\delta = \sqrt{2} - 1$. A larger δ lets in inputs just above $2 + \sqrt{2}$, whose outputs rise above −2.

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SCORING
1 point: the inputs near 3 in the band, $2 < x < 2 + \sqrt{2}$. 1 point: $\delta = \sqrt{2} - 1$, the distance to the nearer end. A decimal alone earns this point only with $\sqrt{2} - 1$ shown.

3. Show that the equation $\sqrt{x + 4} = x - 5$ has a solution strictly between 8 and 9. Give a complete justification with the intermediate value theorem. (3 points)

ANSWER
Let $F(x) = \sqrt{x + 4} - x + 5$, continuous on $[\,8,\ 9\,]$. $F(8) = \sqrt{12} - 3 > 0$ and $F(9) = \sqrt{13} - 4 < 0$, so by the intermediate value theorem $F(c) = 0$, that is $\sqrt{c + 4} = c - 5$, for some c with $8 < c < 9$.

SOLUTION
The solutions are the zeros of $F(x) = \sqrt{x + 4} - x + 5$.

Continuity: a square root is continuous where it is defined, so $\sqrt{x + 4}$ is continuous for $x \geq -4$, which includes all of $[\,8,\ 9\,]$. The polynomial $x - 5$ is continuous everywhere, and a difference of continuous functions is continuous, so F is continuous on $[\,8,\ 9\,]$.

Signs at the ends:

$$F(8) = \sqrt{12} - 3 > 0 \quad \text{because } 12 > 9 = 3^2, \text{ so } \sqrt{12} > 3,$$

$$F(9) = \sqrt{13} - 4 < 0 \quad \text{because } 13 < 16 = 4^2, \text{ so } \sqrt{13} < 4.$$

Conclusion: 0 lies between $F(8)$ and $F(9)$, so by the intermediate value theorem there is a c in $[\,8,\ 9\,]$ with $F(c) = 0$. Neither end is a zero, so $8 < c < 9$, and $\sqrt{c + 4} = c - 5$.

SCORING
1 point: F (or the two sides) named continuous on $[\,8,\ 9\,]$ with a reason (the square root is defined there, a difference of continuous functions). 1 point: $F(8) > 0$ and $F(9) < 0$ shown with the numbers. 1 point: the conclusion, a solution strictly between 8 and 9, citing the intermediate value theorem. A reason from how a graph looks earns no point.

4. Elena's greenhouse controller measures the air temperature C in degrees Celsius and displays it in degrees Fahrenheit, $F(C) = \dfrac{9}{5}C + 32$. At 25 degrees Celsius it displays 77 degrees Fahrenheit. (3 points)

(a) Prove from the definition that $\lim_{C \to 25} F(C) = 77$.

(b) Elena wants the display within 0.9 degrees Fahrenheit of 77. Use your rule from (a) to find how close to 25 degrees Celsius the air temperature must be to guarantee this.

ANSWER
(a) $\delta = \dfrac{5\varepsilon}{9}$ works (proof in the key). (b) Within 0.5 degrees Celsius of 25.

SOLUTION
(a) Scratch work:

$$|F(C) - 77| = \left|\frac{9}{5}C - 45\right| = \frac{9}{5}|C - 25|.$$

The output distance is $\dfrac{9}{5}$ of the input distance, so $\delta = \dfrac{5\varepsilon}{9}$ should work.

Proof. Let $\varepsilon > 0$ and choose $\delta = \dfrac{5\varepsilon}{9}$. If $0 < |C - 25| < \delta$, then

$$|F(C) - 77| = \frac{9}{5}|C - 25| < \frac{9}{5}\delta = \varepsilon.$$

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Since ε was any positive number, the limit is 77.

(b) With $\varepsilon = 0.9$, $\delta = \dfrac{5(0.9)}{9} = 0.5$: temperatures strictly between 24.5 and 25.5 degrees Celsius give displays strictly between 76.1 and 77.9 degrees Fahrenheit.

SCORING

1 point: scratch work, $|F(C) - 77| = \dfrac{9}{5}|C - 25|$. 1 point: $\delta = \dfrac{5\varepsilon}{9}$ chosen for an arbitrary $\varepsilon > 0$, and the forward chain $|F(C) - 77| < \dfrac{9}{5}\delta = \varepsilon$ shown for every allowed C. Scratch work alone, with no forward proof, does not earn this point. 1 point: (b) 0.5 degrees Celsius, with units.

5. This is the scratch work for a proof that $\lim_{x \to -1}(2x^2 - x) = 3$. (3 points)

(a) Factor $2x^2 - x - 3$, and write $|2x^2 - x - 3|$ as $|x + 1|$ times another factor.

(b) Agree that $\delta \leq 1$. Find a number K with $|2x - 3| < K$ for every x with $|x + 1| < 1$.

(c) Write a δ in terms of ε that the proof can use, and give its value for $\varepsilon = 0.14$.

ANSWER

(a) $|2x^2 - x - 3| = |x + 1|\,|2x - 3|$ (b) $K = 7$ (c) $\delta = \min\left(1, \dfrac{\varepsilon}{7}\right)$; for $\varepsilon = 0.14$, $\delta = 0.02$

SOLUTION

(a) $2x^2 - x - 3 = (2x - 3)(x + 1)$, so $|2x^2 - x - 3| = |x + 1|\,|2x - 3|$.

(b) $|x + 1| < 1$ gives $-2 < x < 0$, so $-4 < 2x < 0$ and $-7 < 2x - 3 < -3$. So $|2x - 3| < 7$: $K = 7$.

(c) Under the agreement, $|2x^2 - x - 3| < 7|x + 1|$, which is less than ε once $|x + 1| < \dfrac{\varepsilon}{7}$. Both conditions are needed, so $\delta = \min\left(1, \dfrac{\varepsilon}{7}\right)$. For $\varepsilon = 0.14$, $\delta = \min(1, 0.02) = 0.02$.

SCORING

1 point: (a) the factored form with $|x + 1|$ as one factor. 1 point: (b) $K = 7$ from $-2 < x < 0$. Any larger K shown to work also earns the point, and then (c) is graded with that K. 1 point: (c) $\delta = \min\left(1, \dfrac{\varepsilon}{7}\right)$ with the 1 kept inside the minimum, and $\delta = 0.02$.

6. Let $f(x) = \dfrac{x^2 + 5}{x - 1}$. Then $f(2) = 9$ and $f(5) = \dfrac{15}{2}$. (4 points)

(a) Use the intermediate value theorem to show that $f(c) = 8$ for some c in $[2, 5]$.

(b) Andre says: "7 is not between $f(2)$ and $f(5)$, so by the intermediate value theorem the equation $f(x) = 7$ has no solution in $[2, 5]$." Is his reasoning right? Solve $f(x) = 7$ on $[2, 5]$ to check.

ANSWER

(a) f is continuous on $[2, 5]$ (a quotient of polynomials with $x - 1 \neq 0$ there) and $\dfrac{15}{2} < 8 < 9$, so such a c exists. (b) No: the theorem promises a solution when the target is between the end values and says nothing when it is not. In fact $f(x) = 7$ has two solutions in $[2, 5]$, $x = 3$ and $x = 4$.

SOLUTION

(a) f is a quotient of polynomials, continuous wherever the denominator $x - 1$ is not 0. For $2 \leq x \leq 5$, $x - 1 \geq 1$, so f is continuous on $[2, 5]$. The end values are $f(5) = \dfrac{15}{2} = 7.5$ and $f(2) = 9$, and $7.5 < 8 < 9$. By the intermediate value theorem, $f(c) = 8$ for some c in $[2, 5]$.

(b) The reasoning is wrong. The theorem says: if N is between $f(a)$ and $f(b)$ (and f is continuous on $[a, b]$), then $f(c) = N$ for some c. When 7 is not between $\dfrac{15}{2}$ and 9, that condition fails and the theorem says

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nothing either way; it never says a value is missed. To check, solve on $[2,5]$, where $x-1\neq 0$:

$$\frac{x^2+5}{x-1}=7 \iff x^2+5=7x-7 \iff x^2-7x+12=0 \iff (x-3)(x-4)=0.$$

So $x=3$ or $x=4$, both in $[2,5]$, and indeed $f(3)=\dfrac{14}{2}=7$ and $f(4)=\dfrac{21}{3}=7$. Between the ends the graph dips below both end heights, so $f(x)=7$ has two solutions there.

SCORING

1 point: continuity of f on $[2,5]$ with the reason ($x-1\neq 0$ there). 1 point: $\dfrac{15}{2}<8<9$ with the end values and the conclusion. 1 point: Andre's error named: the theorem gives a solution when the target is between the end values and says nothing when it is not. 1 point: $x=3$ and $x=4$ found by solving, both in $[2,5]$.

7. Isabel uses bisection on $p(x)=x^3+2x^2-6$ over $[1,2]$, where $p(1)=-3$ and $p(2)=10$. Her first three rows are below, with values rounded to four decimal places. (3 points)

a	b	m	$p(m)$
1	2	1.5	1.875
1	1.5	1.25	−0.9219
1	1.25	1.125	−2.0449

(a) Find the first row that is wrong, and say what went wrong.

(b) Correct that row so that the bisection has three correct steps. Give the final interval and its midpoint.

ANSWER

(a) The third row: after $p(1.25)<0$ she should keep $[1.25,1.5]$, but she kept $[1,1.25]$, where p is negative at both ends. (b) Row 3 is $a=1.25$, $b=1.5$, $m=1.375$, $p(1.375)\approx 0.3809$; the final interval is $[1.25,1.375]$ with midpoint 1.3125.

SOLUTION

(a) Rows 1 and 2 are right: $p(1.5)=1.875>0$ while $p(1)<0$, so $[1,1.5]$ is kept; then $p(1.25)\approx -0.9219<0$. Now $p(1.25)$ has the same sign as $p(1)$ and the opposite sign to $p(1.5)$, so the root is trapped in $[1.25,1.5]$: a should move to 1.25. Isabel moved b instead and kept $[1,1.25]$, where $p(1)=-3$ and $p(1.25)$ are both negative, so the theorem promises nothing there. (In fact the root, about 1.34, is not in $[1,1.25]$.)

(b) The correct third row is $a=1.25$, $b=1.5$, $m=1.375$, with

$$p(1.375)=2.599609\ldots+3.78125-6\approx 0.3809>0.$$

The ends of $[1.25,1.375]$ have opposite signs, so that interval is kept. Final interval $[1.25,1.375]$, midpoint 1.3125 (within 0.0625 of the root).

SCORING

1 point: row 3 named as the first wrong row, with the reason (after $p(1.25)<0$ the kept half must be $[1.25,1.5]$; p has the same sign at 1 and 1.25). 1 point: $m=1.375$ and $p(1.375)\approx 0.3809>0$. 1 point: final interval $[1.25,1.375]$ and midpoint 1.3125.

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