Calculus 1 Unit 1 test: Limits and continuity (answer key)

12 questions, 49 points

ANSWERS AT A GLANCE

1. (a) 0 and 0; $\lim_{x\to 3} h(x) = 0$. (b) 8 and 10; the limit does not exist. (c) Every integer except 3. At an integer n the one-sided limits are $(n-1)(n-3)$ and $n(n-3)$, which differ unless $n = 3$; at a number that is not an integer, h follows a single polynomial rule on a window around it.
2. $a = -8$, $b = -20$
3. $-\dfrac{4}{3}$
4. (a) Yes: $A(10) = 60$ and both one-sided limits are 60. (b) No: $\lim_{t\to 40^-} A(t) = 60$ but $\lim_{t\to 40^+} A(t) = 72$, so the limit does not exist and Condition 2 fails. (c) The model has the drone jump from 60 meters to 72 meters at the instant $t = 40$, which a real drone cannot do, so the descent rule is wrong.
5. Continuous on $[-1, 3)$ and $(3, \infty)$ by the combination results, from the right at -1 with $\lim_{x\to -1^+} f(x) = 4 = f(-1)$, and at 3 because $\lim_{x\to 3} f(x) = 24 = f(3)$.
6. (a) $c = 1$; the value $\dfrac{5}{4}$. (b) $c = -1$; the value $\dfrac{5}{4}$. (c) For $c \neq \pm 1$ both discontinuities are infinite. Removable at 2 needs $c = 1$ and removable at -2 needs $c = -1$, and c cannot be both.
7. $a = 2$ and $b = 4$. For any other a the top heads to a nonzero number while the bottom heads to 0, so the left limit is infinite and no b can match it.
8. $y = x$ as $x \to \infty$ and $y = -x$ as $x \to -\infty$.
9. (a) 2 (b) ∞
10. (a) f: 1 from the left, 3 from the right; g: 5 from the left, 3 from the right. Neither $\lim_{x\to 1} f(x)$ nor $\lim_{x\to 1} g(x)$ exists. (b) $f + g$: 6 and 6, so $\lim_{x\to 1}\big[f(x) + g(x)\big] = 6$. fg: 5 and 9, so $\lim_{x\to 1} f(x)\,g(x)$ does not exist. (c) The sum law needs both $\lim f(x)$ and $\lim g(x)$ to exist, and neither does.
11. (a) $f(h) = h^3 - 3h^2 + 1$ is continuous and $f(0) = 1 > 0 > -1 = f(1)$, so a solution lies in $(0, 1)$. (b) Midpoints 0.5, 0.75, 0.625; final interval $[0.625, 0.75]$. (c) $h \approx 0.6875$ decimeters, within 0.0625 decimeters of the true depth.
12. (a) At $x = 1$: $\lim_{x\to 1^-} f(x) = -2$ and $\lim_{x\to 1^+} f(x) = 3$. At $x = 3$: $\lim_{x\to 3^-} f(x) = \infty$ and $\lim_{x\to 3^+} f(x) = -\infty$. (b) At $x = -3$: the limit is 2 and $f(-3) = -1$. At $x = 5$: the limit is -1 and $f(5)$ is undefined. (c) $\lim_{x\to -\infty} f(x) = -\infty$ (the outputs decrease without bound) and $\lim_{x\to \infty} f(x) = 1$.

WORKED SOLUTIONS

1. Let $h(x) = \lfloor x \rfloor\,(x - 3)$. (4 points)

(a) Find $\lim_{x\to 3^-} h(x)$ and $\lim_{x\to 3^+} h(x)$. Does $\lim_{x\to 3} h(x)$ exist?

(b) Find $\lim_{x\to 5^-} h(x)$ and $\lim_{x\to 5^+} h(x)$. Does $\lim_{x\to 5} h(x)$ exist?

(c) Find every number a at which $\lim_{x\to a} h(x)$ does not exist, and explain why the limit exists everywhere else.

ANSWER

(a) 0 and 0; $\lim_{x\to 3} h(x) = 0$. (b) 8 and 10; the limit does not exist. (c) Every integer except 3. At an integer n the one-sided limits are $(n-1)(n-3)$ and $n(n-3)$, which differ unless $n = 3$; at a number that is not an integer, h follows a single polynomial rule on a window around it.

SOLUTION

On $n \le x < n + 1$, with n an integer, $\lfloor x \rfloor = n$ and $h(x) = n(x - 3)$, a polynomial.

(a) Just left of 3, $\lfloor x \rfloor = 2$ and $h(x) = 2(x - 3)$, which heads to 0. Just right of 3, $h(x) = 3(x - 3)$, which heads to 0. Both sides give 0, so $\lim_{x\to 3} h(x) = 0$, even though $\lfloor x \rfloor$ itself jumps at 3: the factor $x - 3$ shrinks both pieces to 0.

(b) Just left of 5, $h(x) = 4(x - 3)$, which heads to $4 \cdot 2 = 8$. Just right, $h(x) = 5(x - 3)$, which heads to 10. The sides differ, so the limit does not exist.

(c) At an integer n, the left side uses $\lfloor x \rfloor = n - 1$ and the right side uses n, so

$$\lim_{x\to n^-} h(x) = (n-1)(n-3), \qquad \lim_{x\to n^+} h(x) = n(n-3)$$

These differ by $n - 3$, so they agree only at $n = 3$. The limit therefore fails at every integer other than 3. At a number a that is not an integer, a small window around a lies inside one interval $n < x < n + 1$, where $h(x) = n(x - 3)$; by locality $\lim_{x\to a} h(x) = n(a - 3)$ exists.

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SCORING
1 point: (a) both one-sided limits 0 and the limit 0. 1 point: (b) 8 and 10, does not exist. 1 point: (c) the one-sided limits at a general integer n, $(n-1)(n-3)$ and $n(n-3)$. 1 point: (c) conclusion "every integer except 3" with the reason that non-integers sit in a window with one rule. An answer read from a sketch or a table without the floor's value on each side earns no point for (c).

2. Find the constants a and b for which

$$\lim_{x \to -2} \frac{x^2 + ax + b}{x^2 - 4} = 3.$$

Explain why the numerator must be 0 at $x = -2$. (4 points)

ANSWER
$a = -8, b = -20$

SOLUTION
The denominator is 0 at $x = -2$. If the numerator approached a number other than 0 there, the quotient would be that number divided by numbers near 0, its size would grow without bound, and the limit could not be 3. So the numerator must be 0 at -2:

$$4 - 2a + b = 0, \quad \text{so} \quad b = 2a - 4.$$

Then $x + 2$ is a factor of the numerator (factor theorem): $x^2 + ax + 2a - 4 = (x+2)(x+a-2)$, since $(x+2)(x+a-2) = x^2 + ax + 2a - 4$. With $x^2 - 4 = (x+2)(x-2)$, for $x \neq -2$ the quotient is $\dfrac{x+a-2}{x-2}$, so

$$\lim_{x \to -2} \frac{x^2 + ax + b}{x^2 - 4} = \frac{-2 + a - 2}{-4} = \frac{a-4}{-4}.$$

Setting this equal to 3: $a - 4 = -12$, so $a = -8$ and $b = 2(-8) - 4 = -20$.
Check: $\dfrac{x^2 - 8x - 20}{x^2 - 4} = \dfrac{(x+2)(x-10)}{(x+2)(x-2)}$, and $\dfrac{-12}{-4} = 3$.

SCORING
1 point for explaining that a nonzero numerator over a denominator near 0 cannot give the limit 3, so the numerator is 0 at -2; 1 point for $b = 2a - 4$; 1 point for factoring and canceling to get the limit $\dfrac{a-4}{-4}$ in terms of a; 1 point for $a = -8$ and $b = -20$.

3. Find $\lim_{x \to 0} \dfrac{\sin(x^2 + 4x)}{\tan(x^2 - 3x)}$. Name each limit result you use. (4 points)

ANSWER
$-\dfrac{4}{3}$

SOLUTION
Substituting gives $\dfrac{0}{0}$. Let $u = x^2 + 4x = x(x+4)$ and $w = x^2 - 3x = x(x-3)$. As $x \to 0$, both u and w tend to 0, and for $0 < |x| < 3$ neither equals 0. So by rescaling $\dfrac{\sin u}{u} \to 1$ and $\dfrac{\tan w}{w} \to 1$. Write

$$\frac{\sin u}{\tan w} = \frac{\sin u}{u} \cdot \frac{w}{\tan w} \cdot \frac{u}{w}$$

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and, canceling the common factor x,

$$\frac{u}{w} = \frac{x+4}{x-3} \to \frac{4}{-3}$$

By the product and quotient laws,

$$\lim_{x \to 0} \frac{\sin(x^2+4x)}{\tan(x^2-3x)} = 1 \cdot 1 \cdot \frac{4}{-3} = -\frac{4}{3}$$

SCORING

1 point for naming $u = x^2+4x$ and $w = x^2-3x$ and noting both tend to 0 without equaling 0; 1 point for the rewrite into $\frac{\sin u}{u}$, $\frac{w}{\tan w}$ and $\frac{u}{w}$; 1 point for $\frac{u}{w} \to \frac{4}{-3}$ by canceling x; 1 point for the answer $-\frac{4}{3}$.

4. Diego models the height of a delivery drone, in meters, t seconds after takeoff. The drone climbs, hovers, then comes down:

$$A(t) = \begin{cases} \frac{3}{5}t^2 & \text{if } 0 \le t < 10 \\ 60 & \text{if } 10 \le t \le 40 \\ \frac{(100-t)^2}{50} & \text{if } 40 < t \le 100 \end{cases}$$

(4 points)

(a) Is A continuous at $t = 10$? Check each condition.

(b) Is A continuous at $t = 40$? Check each condition.

(c) A real drone's height cannot change by a jump. What does your answer to (b) say about Diego's model?

ANSWER

(a) Yes: $A(10) = 60$ and both one-sided limits are 60. (b) No: $\lim_{t \to 40^-} A(t) = 60$ but $\lim_{t \to 40^+} A(t) = 72$, so the limit does not exist and Condition 2 fails. (c) The model has the drone jump from 60 meters to 72 meters at the instant $t = 40$, which a real drone cannot do, so the descent rule is wrong.

SOLUTION

(a) Condition 1: $A(10) = 60$ from the middle rule. From the left, $\lim_{t \to 10^-} \frac{3}{5}t^2 = \frac{3}{5} \cdot 100 = 60$ (a polynomial). From the right, $A(t) = 60$, so the right-hand limit is 60. The one-sided limits agree, so $\lim_{t \to 10} A(t) = 60$ (Condition 2), and it equals $A(10) = 60$ (Condition 3). A is continuous at 10.

(b) Condition 1: $A(40) = 60$ from the middle rule. From the left the rule is the constant 60, so the left-hand limit is 60. From the right, $\lim_{t \to 40^+} \frac{(100-t)^2}{50} = \frac{60^2}{50} = \frac{3{,}600}{50} = 72$. The one-sided limits differ, so $\lim_{t \to 40} A(t)$ does not exist and Condition 2 fails. A is not continuous at 40.

(c) Just after $t = 40$ the model puts the drone near 72 meters, while just before it was at 60 meters: a rise of 12 meters in no time at all. A real drone's height changes continuously, so the model is wrong at $t = 40$; the descent rule should start at 60 meters.

SCORING

1 point: (a) value and both one-sided limits at 10 equal 60, so continuous. 1 point: (b) one-sided limits at 40 found as 60 and 72. 1 point: (b) concludes the limit does not exist, so Condition 2 fails and A is not continuous at 40. 1 point: (c) interprets the break in context: a 12 meter jump in height at $t = 40$ that a real drone cannot make, so the model is wrong there.

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5. Let $f(x) = \dfrac{x^2 - 9}{\sqrt{x+1} - 2}$ for $x \geq -1$ with $x \neq 3$, and $f(3) = 24$. Show that f is continuous on $[-1, \infty)$.

(4 points)

ANSWER

Continuous on $[-1, 3)$ and $(3, \infty)$ by the combination results, from the right at -1 with $\lim_{x \to -1^+} f(x) = 4 = f(-1)$, and at 3 because $\lim_{x \to 3} f(x) = 24 = f(3)$.

SOLUTION

Away from 3: the top is a polynomial. $\sqrt{x+1}$ is the square root of a polynomial with values in $[0, \infty)$ for $x \geq -1$, so it is continuous on $[-1, \infty)$, and so is $\sqrt{x+1} - 2$. The bottom is 0 only where $\sqrt{x+1} = 2$, that is at $x = 3$. By the quotient result f is continuous on $[-1, 3)$ and on $(3, \infty)$.

At -1 (the included end): top and bottom are both continuous from the right there and the bottom is $-2 \neq 0$, so

$$\lim_{x \to -1^+} f(x) = \frac{1 - 9}{\sqrt{0} - 2} = \frac{-8}{-2} = 4 = f(-1)$$

At 3 (inside the interval, so both sides): substitution gives $\dfrac{0}{0}$. For $x \neq 3$, multiply top and bottom by $\sqrt{x+1} + 2$:

$$\frac{(x^2 - 9)\left(\sqrt{x+1} + 2\right)}{(x+1) - 4} = \frac{(x-3)(x+3)\left(\sqrt{x+1} + 2\right)}{x - 3} = (x+3)\left(\sqrt{x+1} + 2\right)$$

so $\lim_{x \to 3} f(x) = 6 \cdot (2 + 2) = 24 = f(3)$, and f is continuous at 3.

Continuous on $[-1, 3)$, at 3 and on $(3, \infty)$, f is continuous on $[-1, \infty)$.

SCORING

1 point for continuity away from 3 by the root, difference and quotient results, with 3 the only zero of the bottom; 1 point for continuity from the right at -1 with the value 4 checked; 1 point for $\lim_{x \to 3} f(x) = 24$ by rationalizing and factoring; 1 point for comparing it with $f(3)$ and the conclusion on $[-1, \infty)$. Two-sided continuity at -1 is not asked; an answer that tries to check the left side there loses the second point.

6. For a constant c, let $f(x) = \dfrac{x^2 + cx - 6}{x^2 - 4}$. (4 points)

(a) Find the value of c for which the discontinuity at 2 is removable, and the value at 2 that removes it.

(b) Find the value of c for which the discontinuity at -2 is removable, and the value at -2 that removes it.

(c) Classify both discontinuities for every other value of c, and explain why no c makes both of them removable.

ANSWER

(a) $c = 1$; the value $\dfrac{5}{4}$. (b) $c = -1$; the value $\dfrac{5}{4}$. (c) For $c \neq \pm 1$ both discontinuities are infinite. Removable at 2 needs $c = 1$ and removable at -2 needs $c = -1$, and c cannot be both.

SOLUTION

The bottom is $(x - 2)(x + 2)$, so the only discontinuities are at 2 and -2, whatever c is. As $x \to 2$ the top approaches $4 + 2c - 6 = 2c - 2$, and as $x \to -2$ it approaches $4 - 2c - 6 = -2c - 2$.

If the top approaches a nonzero number while the bottom approaches 0, the quotient runs off to $\pm\infty$ on each side: an infinite discontinuity. So a removable discontinuity is possible only where the top also approaches 0.

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(a) At 2 this needs $2c - 2 = 0$, so $c = 1$. Then $x^2 + x - 6 = (x + 3)(x - 2)$, and for $x \neq \pm 2$, $f(x) = \dfrac{x + 3}{x + 2}$, which approaches $\dfrac{5}{4}$ at 2. The limit exists and $f(2)$ does not: removable, removed by the value $\dfrac{5}{4}$.

(b) At -2 this needs $-2c - 2 = 0$, so $c = -1$. Then $x^2 - x - 6 = (x - 3)(x + 2)$, and $f(x) = \dfrac{x - 3}{x - 2}$ for $x \neq \pm 2$, which approaches $\dfrac{-5}{-4} = \dfrac{5}{4}$ at -2: removable, removed by the value $\dfrac{5}{4}$.

(c) For $c \neq \pm 1$ the top approaches a nonzero number at both 2 and -2, so both discontinuities are infinite. Both removable would need $c = 1$ and $c = -1$ at once, which is impossible. (With $c = 1$ the top approaches -4 at -2, and with $c = -1$ it approaches -4 at 2, so in each case the other discontinuity is infinite.)

SCORING

1 point: (a) $c = 1$ and the value $\dfrac{5}{4}$, from the top being 0 at 2 and the canceled form. 1 point: (b) $c = -1$ and the value $\dfrac{5}{4}$. 1 point: (c) both infinite for $c \neq \pm 1$, because the top tends to a nonzero number while the bottom tends to 0. 1 point: (c) no c works for both, since the conditions $c = 1$ and $c = -1$ conflict.

7. Find the constants a and b that make f continuous at every number, and explain why only one value of a can work.

$$f(x) = \begin{cases} \dfrac{x^2 + ax - 15}{x - 3}, & x < 3 \\ bx - 4, & x \geq 3 \end{cases}$$

(4 points)

ANSWER

$a = 2$ and $b = 4$. For any other a the top heads to a nonzero number while the bottom heads to 0, so the left limit is infinite and no b can match it.

SOLUTION

For $x < 3$ the left rule is a rational function whose bottom is not 0, and the right rule is a line, so only $x = 3$ matters. The right rule includes 3: $f(3) = 3b - 4$, a number for every b.

Why a is forced. As $x \to 3^-$ the bottom $x - 3$ heads to 0 and the top heads to $9 + 3a - 15 = 3a - 6$. If $3a - 6 \neq 0$, the outputs grow without bound, the left limit is not a number, and no choice of b can join the pieces. So the top must head to 0: $3a - 6 = 0$, $a = 2$.

The left limit. With $a = 2$ the top is $x^2 + 2x - 15 = (x + 5)(x - 3)$, so for $x < 3$ the quotient is $x + 5$, which tends to 8.

Join. $3b - 4 = 8$, so $b = 4$. Check: 4(3) − 4 = 8.

SCORING

1 point: explains that the top must head to 0 (otherwise the left limit is infinite) and finds $a = 2$. 1 point: factors $(x + 5)(x - 3)$ and finds the left limit 8. 1 point: sets $3b - 4 = 8$ with the value from the right rule. 1 point: $b = 4$.

8. Let $f(x) = \dfrac{x^2 + 1}{\sqrt{x^2 + 4}}$. Use limits to find the oblique asymptote of the graph of f at each end. (Long division does not apply, since f is not a rational function.) (4 points)

ANSWER

$y = x$ as $x \to \infty$ and $y = -x$ as $x \to -\infty$.

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SOLUTION

Right end. For $x > 0$, $\sqrt{x^2+4} = x\sqrt{1+\frac{4}{x^2}}$. Then

$$m = \lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{x^2+1}{x^2\sqrt{1+\frac{4}{x^2}}} = \lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{\sqrt{1+\frac{4}{x^2}}} = 1.$$

Next, $f(x)-x = \dfrac{x^2+1-x\sqrt{x^2+4}}{\sqrt{x^2+4}}$. In the top both terms grow, so multiply the top by its conjugate:

$$x^2+1-x\sqrt{x^2+4} = \frac{(x^2+1)^2-x^2(x^2+4)}{x^2+1+x\sqrt{x^2+4}} = \frac{1-2x^2}{x^2+1+x\sqrt{x^2+4}}.$$

Dividing by x^2, this is $\dfrac{\frac{1}{x^2}-2}{1+\frac{1}{x^2}+\sqrt{1+\frac{4}{x^2}}} \to \dfrac{-2}{2} = -1$. Also $\dfrac{1}{\sqrt{x^2+4}} = \dfrac{1}{x}\cdot\dfrac{1}{\sqrt{1+\frac{4}{x^2}}} \to 0\cdot 1 = 0$. By the product law, $b = \lim_{x\to\infty}\big(f(x)-x\big) = (-1)(0) = 0$. The asymptote on the right is $y = x$.

Left end. For $x < 0$, $\sqrt{x^2+4} = -x\sqrt{1+\frac{4}{x^2}}$, so $\dfrac{f(x)}{x} = -\dfrac{1+\frac{1}{x^2}}{\sqrt{1+\frac{4}{x^2}}} \to -1$, and $m = -1$. Since $f(-x) = f(x)$, put $u = -x$: $f(x)+x = f(u)-u$, and $u \to \infty$ as $x \to -\infty$, so $b = \lim_{x\to-\infty}\big(f(x)+x\big) = 0$ by the right-end work. The asymptote on the left is $y = -x$.

SCORING

1 point for $m = 1$ on the right with $\sqrt{x^2+4} = x\sqrt{1+\frac{4}{x^2}}$; 1 point for $b = 0$ on the right with a valid rewrite (conjugate, or another correct route); 1 point for $m = -1$ on the left using $|x| = -x$; 1 point for $b = 0$ on the left and both lines stated, $y = x$ and $y = -x$.

9. Find each limit.

(a) $\displaystyle\lim_{x\to\infty}\frac{\ln\left(x^6+4^x\right)}{\ln\left(x^6+2^x\right)}$ (b) $\displaystyle\lim_{x\to\infty}\frac{x^6+4^x}{x^6+2^x}$ (4 points)

ANSWER

(a) 2 (b) ∞

SOLUTION

(a) Factor the fastest term out of each logarithm. For $x > 0$,

$$\ln\left(x^6+4^x\right) = \ln\left(4^x\left(1+\frac{x^6}{4^x}\right)\right) = x\ln 4 + \ln\left(1+\frac{x^6}{4^x}\right),$$

and in the same way $\ln\left(x^6+2^x\right) = x\ln 2 + \ln\left(1+\dfrac{x^6}{2^x}\right)$. Divide top and bottom by x:

$$\frac{\ln 4 + \frac{1}{x}\ln\left(1+\frac{x^6}{4^x}\right)}{\ln 2 + \frac{1}{x}\ln\left(1+\frac{x^6}{2^x}\right)}.$$

By the ladder $\dfrac{x^6}{4^x} \to 0$ and $\dfrac{x^6}{2^x} \to 0$, so by continuity of ln at 1 each logarithm goes to $\ln 1 = 0$, and multiplied by $\dfrac{1}{x} \to 0$ it still goes to 0. The bottom goes to $\ln 2 \neq 0$, so the limit is $\dfrac{\ln 4}{\ln 2} = \dfrac{2\ln 2}{\ln 2} = 2$.

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(b) Divide top and bottom by 2^x:

$$\frac{\dfrac{x^6}{2^x} + 2^x}{\dfrac{x^6}{2^x} + 1}.$$

The bottom goes to 0 + 1 = 1 and the top grows without bound, so the limit is ∞. Taking logarithms squeezed the two functions into the same rate, but the functions themselves are on different rungs.

SCORING

1 point for factoring 4^x and 2^x out of the logarithms and splitting each into $x \ln(\text{base})$ plus a remainder; 1 point for showing each remainder over x goes to 0 by the ladder; 1 point for (a) 2; 1 point for (b) ∞ with a reason.

10. Let

$$f(x) = \begin{cases} x, & x < 1 \\ x + 2, & x \geq 1 \end{cases} \qquad g(x) = \begin{cases} 2x + 3, & x < 1 \\ x^2 + 2, & x \geq 1 \end{cases}$$

(a) Find $\lim_{x \to 1^-}$ and $\lim_{x \to 1^+}$ of $f(x)$ and of $g(x)$. Does $\lim_{x \to 1} f(x)$ exist? Does $\lim_{x \to 1} g(x)$? (4 points)

(b) Use the limit laws for one-sided limits to find $\lim_{x \to 1^-}$ and $\lim_{x \to 1^+}$ of $f(x) + g(x)$ and of $f(x)\,g(x)$. Which of $\lim_{x \to 1} \big[f(x) + g(x)\big]$ and $\lim_{x \to 1} f(x)\,g(x)$ exist?

(c) Explain why $\lim_{x \to 1} \big[f(x) + g(x)\big] = \lim_{x \to 1} f(x) + \lim_{x \to 1} g(x)$ cannot be written here.

ANSWER

(a) f: 1 from the left, 3 from the right; g: 5 from the left, 3 from the right. Neither $\lim_{x \to 1} f(x)$ nor $\lim_{x \to 1} g(x)$ exists. (b) $f + g$: 6 and 6, so $\lim_{x \to 1} \big[f(x) + g(x)\big] = 6$. fg: 5 and 9, so $\lim_{x \to 1} f(x)\,g(x)$ does not exist. (c) The sum law needs both $\lim f(x)$ and $\lim g(x)$ to exist, and neither does.

SOLUTION

(a) For $x < 1$ the first rules are used, and they are polynomials: $\lim_{x \to 1^-} f(x) = 1$ and $\lim_{x \to 1^-} g(x) = 2 + 3 = 5$. For $x > 1$ the second rules are used: $\lim_{x \to 1^+} f(x) = 1 + 2 = 3$ and $\lim_{x \to 1^+} g(x) = 1 + 2 = 3$. For each function the two one-sided limits differ ($1 \neq 3$ and $5 \neq 3$), so neither two-sided limit exists.

(b) The laws hold for one-sided limits, and each one-sided limit above is a number. From the left: $f + g \to 1 + 5 = 6$ and $fg \to 1 \cdot 5 = 5$. From the right: $f + g \to 3 + 3 = 6$ and $fg \to 3 \cdot 3 = 9$. The sum has equal one-sided limits, so $\lim_{x \to 1} \big[f(x) + g(x)\big] = 6$. The product has $5 \neq 9$, so $\lim_{x \to 1} f(x)\,g(x)$ does not exist.

(c) The sum law turns two limits that exist into a third. Here $\lim_{x \to 1} f(x)$ and $\lim_{x \to 1} g(x)$ do not exist, so the right side is not a sum of two numbers, even though the left side is 6. The sum has a limit because the jumps cancel: f jumps up by 2 at 1 and g jumps down by 2.

SCORING

1 point for all four one-sided limits in (a) and the conclusion that neither two-sided limit exists; 1 point for the sum's one-sided limits 6 and 6 and $\lim_{x \to 1} \big[f(x) + g(x)\big] = 6$; 1 point for the product's one-sided limits 5 and 9 and that its limit does not exist; 1 point for (c) naming the hypothesis that both limits must exist.

11. Amara's fishing float is a wooden ball of radius 1 decimeter. In a made-up model for its wood, the float rests with its lowest point h decimeters below the water surface, where h solves the equation below.

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$$h^3 - 3h^2 + 1 = 0$$

(a) Use the intermediate value theorem to show that the equation has a solution with $0 < h < 1$. (4 points)

(b) Take three bisection steps on $[0, 1]$. Use a calculator and round each value to four decimal places.

(c) Give your estimate of h (the last midpoint) and the most it can differ from the true depth.

ANSWER

(a) $f(h) = h^3 - 3h^2 + 1$ is continuous and $f(0) = 1 > 0 > -1 = f(1)$, so a solution lies in (0, 1). (b) Midpoints 0.5, 0.75, 0.625; final interval $[0.625, 0.75]$. (c) $h \approx 0.6875$ decimeters, within 0.0625 decimeters of the true depth.

SOLUTION

(a) Let $f(h) = h^3 - 3h^2 + 1$. It is a polynomial, so it is continuous on $[0, 1]$. $f(0) = 1 > 0$ and $f(1) = 1 - 3 + 1 = -1 < 0$. Since 0 is between $f(1)$ and $f(0)$, the intermediate value theorem gives c in $[0, 1]$ with $f(c) = 0$; neither end is a zero, so $0 < c < 1$.

(b) f is positive at the left end and negative at the right end.

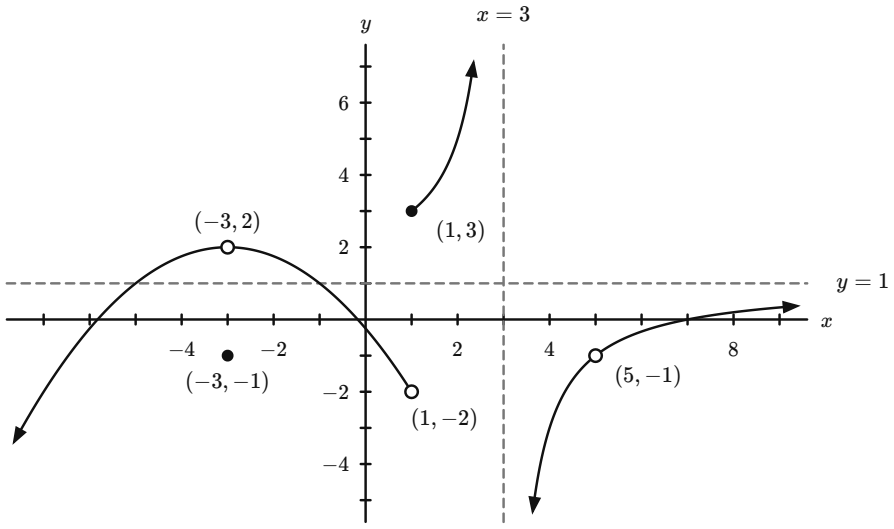
step	a	b	m	$f(m)$	**kept**
1	0	1	0.5	0.3750	$[0.5, 1]$
2	0.5	1	0.75	−0.2656	$[0.5, 0.75]$
3	0.5	0.75	0.625	0.0723	$[0.625, 0.75]$

(c) The depth lies in $[0.625, 0.75]$, of length $\frac{1}{8}$. Its midpoint 0.6875 is within $\frac{1}{16} = 0.0625$ of the true depth. So $h \approx 0.6875$ decimeters, with error at most 0.0625 decimeters. (In fact $h \approx 0.6527$.)

SCORING

1 point: f named continuous on $[0, 1]$ because it is a polynomial. 1 point: $f(0) = 1 > 0$ and $f(1) = -1 < 0$, with the conclusion of a solution in (0, 1). 1 point: three correct midpoints with the signs of f and the kept intervals, ending at $[0.625, 0.75]$. 1 point: estimate $h \approx 0.6875$ and error at most 0.0625, with decimeters stated at least once in (c).

12. The graph of f is drawn below. The tick marks are 1 unit apart, the dashed lines are $x = 3$ and $y = 1$, and to the left of $x = 1$ the graph is part of a parabola. The arrows show the graph going on in the same way. (5 points)



(a) Find every a with $-7 < a < 9$ at which $\lim_{x \to a} f(x)$ does not exist. For each one, give or describe both one-sided limits.

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(b) Find every a with $-7 < a < 9$ at which $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$, counting the case where $f(a)$ is undefined. Give the limit and $f(a)$ for each.

(c) Find $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$.

ANSWER

(a) At $x = 1$: $\lim_{x \to 1^-} f(x) = -2$ and $\lim_{x \to 1^+} f(x) = 3$. At $x = 3$: $\lim_{x \to 3^-} f(x) = \infty$ and $\lim_{x \to 3^+} f(x) = -\infty$. (b) At $x = -3$: the limit is 2 and $f(-3) = -1$. At $x = 5$: the limit is −1 and $f(5)$ is undefined. (c) $\lim_{x \to -\infty} f(x) = -\infty$ (the outputs decrease without bound) and $\lim_{x \to \infty} f(x) = 1$.

SOLUTION

Slide along the graph toward each special input from both sides; everywhere else the graph is one unbroken stroke through its own point, so the limit exists and equals the value.

(a) At $x = 1$ the parabola falls to the open circle (1, −2) from the left, and the next piece starts at the closed dot (1, 3) on the right. The one-sided limits −2 and 3 differ, so there is no limit: a jump. At $x = 3$ the curve climbs past every height on the left and comes up from below every height on the right, so $\lim_{x \to 3^-} f(x) = \infty$ and $\lim_{x \to 3^+} f(x) = -\infty$; the sides go opposite ways, so the limit does not exist and not even $\lim_{x \to 3} f(x) = \infty$ can be written.

(b) At $x = -3$ the parabola heads into the open circle (−3, 2) from both sides, so the limit is 2, but the closed dot below gives $f(-3) = -1$: a moved point. At $x = 5$ the curve heads into the open circle (5, −1) from both sides, so the limit is −1, and no dot sits above 5, so $f(5)$ is undefined: a hole.

(c) A parabola opening down falls past every height far to the left, so $\lim_{x \to -\infty} f(x) = -\infty$, which says the outputs decrease without bound (no limit). Far to the right the curve settles onto the dashed line $y = 1$, so $\lim_{x \to \infty} f(x) = 1$.

SCORING

2 points: (a) both $x = 1$ and $x = 3$ with their one-sided limits (1 point each); deduct 1 for listing an extra point. 2 points: (b) $x = -3$ with limit 2 and $f(-3) = -1$, and $x = 5$ with limit −1 and $f(5)$ undefined (1 point each); deduct 1 for listing an extra point. 1 point: (c) both limits at infinity.

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