Calculus 1 Unit 1 worksheet: The intermediate value theorem (answer key)

ANSWERS AT A GLANCE

1. (a) At least three zeros: one in each of (−2, 0), (0, 3) and (3, 6). (b) Yes, in (3, 6). (c) Not necessarily: 5 is not between any two values in the table, so the theorem says nothing.
2. $k = 4$. With $F(x) = x^3 + 4x - 89$, $F(4) = -9 < 0 < 56 = F(5)$, and F is increasing, so the solution in (4, 5) is the only real one.
3. (a) $f(1) = 3 > 0 > -13 = f(2)$ and f is continuous, so a zero lies in (1, 2). (b) 4 steps. (c) Midpoints 1.5, 1.25, 1.375, 1.4375; the zero lies in $[1.375, 1.4375]$, and the estimate is 1.40625, within $\frac{1}{32} = 0.03125$ of the zero.
4. (a) $f(1) \approx 1.56 > 0$ and $f(2) \approx -2.19 < 0$. (b) Midpoints 1.5, 1.75, 1.625; the kept intervals are $[1.5, 2]$, $[1.5, 1.75]$, $[1.5, 1.625]$. (c) No zero. The intervals close in on $\frac{\pi}{2} \approx 1.5708$, where $\tan x$ is not defined, so f is not continuous on $[1, 2]$ and the theorem does not apply.
5. Apply the intermediate value theorem to $g(d) = E(d + 5) - E(d)$ on $[0, 5]$: $g(5) = -g(0)$, so either $g(0) = 0$ or $g(0)$ and $g(5)$ have opposite signs.

WORKED SOLUTIONS

1. A function f is continuous on $[-2, 6]$. All that is known about it is this table.

x	−2	0	3	6
$f(x)$	3	−1	4	−5

(a) What is the least number of zeros f must have in $[-2, 6]$? Give an interval for each one.

(b) Must the equation $f(x) = -3$ have a solution in $[-2, 6]$? If so, in which interval between table entries?

(c) Must the equation $f(x) = 5$ have a solution in $[-2, 6]$? Explain.

ANSWER

(a) At least three zeros: one in each of (−2, 0), (0, 3) and (3, 6). (b) Yes, in (3, 6). (c) Not necessarily: 5 is not between any two values in the table, so the theorem says nothing.

SOLUTION

Since f is continuous on $[-2, 6]$, it is continuous on each of the closed intervals $[-2, 0]$, $[0, 3]$ and $[3, 6]$, so the intermediate value theorem applies on each.

(a) $f(-2) = 3 > 0 > -1 = f(0)$, so f has a zero in (−2, 0). $f(0) = -1 < 0 < 4 = f(3)$, so f has a zero in (0, 3). $f(3) = 4 > 0 > -5 = f(6)$, so f has a zero in (3, 6). These open intervals do not overlap, so the three zeros are different numbers: f has at least three zeros. (It may have more; the theorem gives at least one in each interval, not exactly one.)

(b) $f(3) = 4 > -3 > -5 = f(6)$, so by the theorem $f(c) = -3$ for some c in (3, 6). On the other two intervals −3 is not between the end values (−3 < −1), so the theorem promises a solution only in (3, 6).

(c) Not necessarily. The largest value in the table is 4, so 5 is not between the values at the ends of any interval, and the theorem says nothing. For example, the function whose graph is the broken line through (−2, 3), (0, −1), (3, 4) and (6, −5) is continuous, matches the table, and never goes above 4, so it never equals 5. (Another continuous function with the same table could rise above 5 between 0 and 3, so the answer is not "never" either.)

2. Find the integer k for which the equation $x^3 + 4x = 89$ has a solution strictly between k and $k + 1$. Justify your answer with the intermediate value theorem, and explain why the equation has no other real solution.

ANSWER

$k = 4$. With $F(x) = x^3 + 4x - 89$, $F(4) = -9 < 0 < 56 = F(5)$, and F is increasing, so the solution in (4, 5) is the only real one.

SOLUTION

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Let $F(x) = x^3 + 4x - 89$; the solutions of the equation are the zeros of F. F is a polynomial, so it is continuous everywhere, in particular on $[\,4,\ 5\,]$.

Try whole numbers: $4^3 + 4 \cdot 4 = 80$ is less than 89, and $5^3 + 4 \cdot 5 = 145$ is more. So

$$F(4) = 80 - 89 = -9 < 0, \qquad F(5) = 145 - 89 = 56 > 0.$$

By the intermediate value theorem, $F(c) = 0$ for some c in $[\,4,\ 5\,]$, and c is not an end because F is not 0 there. So $4 < c < 5$ and $k = 4$.

No other solution: if $u < v$, then $u^3 < v^3$ (cubing keeps order) and $4u < 4v$, so $F(u) < F(v)$. F is increasing, so it takes the value 0 at most once.

3. Let $f(x) = 5 - x - x^4$.

(a) Show that f has a zero strictly between 1 and 2.

(b) Find the fewest bisection steps, starting from $[1, 2]$, that guarantee the midpoint of the last interval is within 0.05 of the zero.

(c) Carry out exactly that many steps. Use a calculator and round each value of f to four decimal places. For each step give the midpoint, the value of f there, and the interval you keep. Then give the final interval and your estimate of the zero.

ANSWER

(a) $f(1) = 3 > 0 > -13 = f(2)$ and f is continuous, so a zero lies in $(1, 2)$. (b) 4 steps. (c) Midpoints 1.5, 1.25, 1.375, 1.4375; the zero lies in $[1.375, 1.4375]$, and the estimate is 1.40625, within $\dfrac{1}{32} = 0.03125$ of the zero.

SOLUTION

(a) f is a polynomial, so it is continuous on $[1, 2]$. $f(1) = 5 - 1 - 1 = 3 > 0$ and $f(2) = 5 - 2 - 16 = -13 < 0$. The signs are opposite, so by the intermediate value theorem f has a zero c with $1 < c < 2$.

(b) After n steps from an interval of length 1, the midpoint is within $\dfrac{1}{2^{n+1}}$ of the zero. We need $\dfrac{1}{2^{n+1}} \le 0.05$, that is $2^{n+1} \ge 20$. Since $2^4 = 16$ is too small and $2^5 = 32$ is enough, $n + 1 = 5$ and $n = 4$.

(c) Here f is positive at the left end and negative at the right end, so keep the half whose ends still have opposite signs: if $f(m) > 0$, a moves to m; if $f(m) < 0$, b moves to m.

step	a	b	m	$f(m)$	**kept**
1	1	2	1.5	−1.5625	$[1, 1.5]$
2	1	1.5	1.25	1.3086	$[1.25, 1.5]$
3	1.25	1.5	1.375	0.0505	$[1.375, 1.5]$
4	1.375	1.5	1.4375	−0.7075	$[1.375, 1.4375]$

The zero lies in $[1.375, 1.4375]$, an interval of length $\dfrac{1}{16}$. Its midpoint 1.40625 is within half of that, $\dfrac{1}{32} = 0.03125$, of the zero, which meets the goal of 0.05. (In fact the zero is about 1.3794.)

4. Kevin runs bisection on $f(x) = \tan x$ over $[1, 2]$, with x in radians. Use a calculator in radian mode and round each value to two decimal places.

(a) Show that $f(1)$ and $f(2)$ have opposite signs.

(b) Take three bisection steps. For each step give the midpoint, the value of f there, and the interval kept.

(c) Does $\tan x$ have a zero in $[1, 2]$? What number are Kevin's intervals closing in on? Explain why this does not contradict the intermediate value theorem.

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ANSWER

(a) $f(1) \approx 1.56 > 0$ and $f(2) \approx -2.19 < 0$. (b) Midpoints 1.5, 1.75, 1.625; the kept intervals are $[1.5, 2]$, $[1.5, 1.75]$, $[1.5, 1.625]$. (c) No zero. The intervals close in on $\dfrac{\pi}{2} \approx 1.5708$, where $\tan x$ is not defined, so f is not continuous on $[1, 2]$ and the theorem does not apply.

SOLUTION

(a) $f(1) = \tan 1 \approx 1.56 > 0$ and $f(2) = \tan 2 \approx -2.19 < 0$.

(b)

step	interval	m	$f(m)$	kept
1	$[1, 2]$	1.5	14.10	$[1.5, 2]$
2	$[1.5, 2]$	1.75	−5.52	$[1.5, 1.75]$
3	$[1.5, 1.75]$	1.625	−18.43	$[1.5, 1.625]$

Each time the kept half is the one whose ends have opposite signs.

(c) $\tan x = \dfrac{\sin x}{\cos x}$ is 0 only where $\sin x = 0$, at multiples of π; the nearest are 0 and $\pi \approx 3.14$, both outside $[1, 2]$. So $\tan x$ has no zero in $[1, 2]$.

Every kept interval has $\tan x$ positive at its left end and negative at its right end. On $[1, 2]$, $\tan x > 0$ only for $x < \dfrac{\pi}{2}$ and $\tan x < 0$ only for $x > \dfrac{\pi}{2}$, so every left end is below $\dfrac{\pi}{2} \approx 1.5708$ and every right end is above it. The intervals all contain $\dfrac{\pi}{2}$ and their lengths halve, so their ends close in on it. There $\cos x = 0$, so $\tan x$ is undefined, with $\lim_{x \to \pi/2^-} \tan x = \infty$ and $\lim_{x \to \pi/2^+} \tan x = -\infty$. The theorem needs f continuous on the whole closed interval $[1, 2]$, and $\tan x$ is not: it has an infinite break at $\dfrac{\pi}{2}$. With a hypothesis false, the theorem promises nothing, and bisection traps the break instead of a zero.

5. Jasmine runs a loop route 10 km long that starts and ends at the same spot. Let $E(d)$ be her elevation, in meters, after she has run d km. Then E is continuous on $[0, 10]$ and $E(0) = E(10)$. Show that some two points on the loop that are 5 km apart along the route have the same elevation; that is, show that $E(c) = E(c + 5)$ for some c in $[0, 5]$.

ANSWER

Apply the intermediate value theorem to $g(d) = E(d + 5) - E(d)$ on $[0, 5]$: $g(5) = -g(0)$, so either $g(0) = 0$ or $g(0)$ and $g(5)$ have opposite signs.

SOLUTION

Let $g(d) = E(d + 5) - E(d)$ for d in $[0, 5]$. For such d, both d and $d + 5$ lie in $[0, 10]$, where E is continuous; $E(d + 5)$ is a composition of continuous functions and g is a difference of continuous functions, so g is continuous on $[0, 5]$.

At the ends,

$$g(0) = E(5) - E(0), \qquad g(5) = E(10) - E(5) = E(0) - E(5) = -g(0),$$

using $E(10) = E(0)$.

If $g(0) = 0$, then $E(0) = E(5)$ and $c = 0$ works. Otherwise $g(0)$ and $g(5) = -g(0)$ are nonzero with opposite signs, so 0 lies between them, and by the intermediate value theorem $g(c) = 0$ for some c in $(0, 5)$. Then $E(c + 5) = E(c)$: the points c km and $c + 5$ km along the route are 5 km apart and at the same elevation.

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