Calculus 1 Unit 1 worksheet: Continuity at a point (answer key)

ANSWERS AT A GLANCE

1. (a) 3 (b) $\frac{9}{5}$ (c) 4 (d) π (e) −2
2. Left limit 3, right limit 3, value 5. Not continuous: Condition 3 fails.
3. $\lim_{x\to 3} f(x) = \frac{1}{6}$. Not continuous: the limit exists but is not the value, so Condition 3 fails.
4. (a) $u+v$ and uv are continuous at 0; $u-v$ and $\frac{u}{v}$ are not, because Condition 2 fails for both. (b) Those results need both functions continuous at 0, and neither is, so they say nothing here; each combination has to be checked directly.
5. (a) $c > 4$ (b) $c = 4$; the discontinuity is at $x = -2$. (c) At −3 and −1. At −1 the value $r(-1)$ is not defined, so Condition 1 fails, even though $\lim_{x\to -1} r(x) = \frac{1}{2}$ exists.
6. (a) False: for example $a = 0$, $g(x) = x + 1$, and $f(u) = 0$ for $u < 1$, $f(u) = 1$ for $u \geq 1$. (b) True.

WORKED SOLUTIONS

1. Use continuity to find each limit. Name the result that lets you substitute.

(a) $\lim_{x\to -1}(2x^3 - x + 4)$ (b) $\lim_{x\to 2} \frac{x^2+5}{3x-1}$ (c) $\lim_{x\to 5} \sqrt{x^2-9}$

(d) $\lim_{x\to \pi} x \sin\frac{x}{2}$ (e) $\lim_{x\to 3} \sqrt[3]{1-x^2}$

ANSWER

(a) 3 (b) $\frac{9}{5}$ (c) 4 (d) π (e) −2

SOLUTION

(a) A polynomial is continuous at every number, so the limit is the value: $2(-1)^3 - (-1) + 4 = -2 + 1 + 4 = 3$.

(b) A rational function is continuous where its bottom is not 0. Here the bottom at 2 is $3 \cdot 2 - 1 = 5 \neq 0$, so the limit is $\frac{4+5}{5} = \frac{9}{5}$.

(c) Composition: the inside $x^2 - 9$ is a polynomial with value 16 at 5, and $\sqrt{u}$ is continuous at $16 > 0$. The limit is $\sqrt{16} = 4$.

(d) Product: x is continuous at π, and $\sin\frac{x}{2}$ is continuous at π by the composition result ($\frac{x}{2}$ is a polynomial and sin is continuous everywhere). The limit is $\pi \sin\frac{\pi}{2} = \pi \cdot 1 = \pi$.

(e) Composition: the inside $1 - x^2$ has value $1 - 9 = -8$ at 3, and a cube root is continuous at every number, negative ones included. The limit is $\sqrt[3]{-8} = -2$.

2. Let

$$f(x) = \begin{cases} 2x - 1 & \text{if } x < 2 \\ 5 & \text{if } x = 2 \\ x^2 - 1 & \text{if } x > 2 \end{cases}$$

Check each of the three conditions at 2 and decide whether f is continuous at 2.

ANSWER

Left limit 3, right limit 3, value 5. Not continuous: Condition 3 fails.

SOLUTION

Condition 1: $f(2) = 5$, so $f(2)$ is defined.

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For Condition 2, find the one-sided limits from the rules. To the left of 2 the rule is the polynomial $2x - 1$, so $\lim_{x \to 2^-} f(x) = 2 \cdot 2 - 1 = 3$. To the right the rule is $x^2 - 1$, so $\lim_{x \to 2^+} f(x) = 4 - 1 = 3$.

The one-sided limits are equal, so the limit exists and Condition 2 holds. But the limit is not the value, so Condition 3 fails, and f is not continuous at this point.

3. Let $f(x) = \dfrac{\sqrt{x+6} - 3}{x - 3}$ for $x \neq 3$, and $f(3) = \dfrac{1}{3}$. Find $\lim_{x \to 3} f(x)$, and decide whether f is continuous at 3. If it is not, name the condition that fails.

ANSWER

$\lim_{x \to 3} f(x) = \dfrac{1}{6}$. Not continuous: the limit exists but is not the value, so Condition 3 fails.

SOLUTION

Condition 1 holds, since $f(3) = \dfrac{1}{3}$ is defined.

For the limit, substituting $x = 3$ into the formula gives $\dfrac{\sqrt{9} - 3}{0} = \dfrac{0}{0}$, so rationalize the numerator. For $x \neq 3$,

$$\frac{\sqrt{x+6} - 3}{x - 3} \cdot \frac{\sqrt{x+6} + 3}{\sqrt{x+6} + 3} = \frac{x + 6 - 9}{(x - 3)\left(\sqrt{x+6} + 3\right)} = \frac{1}{\sqrt{x+6} + 3}$$

because $x + 6 - 9 = x - 3$, which cancels. The last expression is continuous at 3 (a composition inside a quotient whose bottom is not 0), so

$$\lim_{x \to 3} f(x) = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

The limit exists, so Condition 2 holds. The limit is not the value, so Condition 3 fails, and f is not continuous at this point.

4. Let $u(x) = 2$ for $x \geq 0$ and $u(x) = 3$ for $x < 0$. Let $v(x) = 3$ for $x \geq 0$ and $v(x) = 2$ for $x < 0$. Neither u nor v is continuous at 0.

(a) Which of $u + v$, $u - v$, uv and $\dfrac{u}{v}$ are continuous at 0? Check the conditions for each.

(b) Explain why the results for sums, products and quotients cannot settle part (a).

ANSWER

(a) $u + v$ and uv are continuous at 0; $u - v$ and $\dfrac{u}{v}$ are not, because Condition 2 fails for both. (b) Those results need both functions continuous at 0, and neither is, so they say nothing here; each combination has to be checked directly.

SOLUTION

(a) Work out each combination on each side of 0 and at 0.

$u + v$ is 2 + 3 = 5 for $x \geq 0$ and 3 + 2 = 5 for $x < 0$. Both one-sided limits are 5 and the value at 0 is 5: all three conditions hold, so $u + v$ is continuous at 0.

$u - v$ is 2 − 3 = −1 for $x \geq 0$ and 3 − 2 = 1 for $x < 0$. The one-sided limits are −1 and 1, so the limit does not exist: Condition 2 fails.

uv is 2 · 3 = 6 for $x \geq 0$ and 3 · 2 = 6 for $x < 0$, so it is the constant 6 and is continuous at 0.

$\dfrac{u}{v}$ is $\dfrac{2}{3}$ for $x \geq 0$ and $\dfrac{3}{2}$ for $x < 0$. The one-sided limits differ, so Condition 2 fails.

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(b) The sum, difference, product and quotient results start from "if f and g are continuous at a". That hypothesis fails for u and v, so the results give no answer either way. They go one way only, and part (a) shows that both outcomes are possible when the hypothesis fails.

5. Let $r(x) = \dfrac{x+1}{x^2+4x+c}$, where c is a constant.

(a) For which values of c is r continuous at every number?

(b) For which value of c is r discontinuous at exactly one number? Which number?

(c) Take $c = 3$. At which numbers is r discontinuous? Explain why canceling a common factor does not make r continuous at -1.

ANSWER

(a) $c > 4$ (b) $c = 4$; the discontinuity is at $x = -2$. (c) At -3 and -1. At -1 the value $r(-1)$ is not defined, so Condition 1 fails, even though $\lim_{x \to -1} r(x) = \dfrac{1}{2}$ exists.

SOLUTION

r is a rational function, so it is continuous at every number where the bottom is not 0 and discontinuous where the bottom is 0 (Condition 1 fails there). So the question is where $x^2 + 4x + c = 0$.

(a) The bottom has no real zeros exactly when its discriminant is negative: $4^2 - 4c < 0$, so $16 < 4c$ and $c > 4$.

(b) Exactly one real zero means the discriminant is 0: $16 - 4c = 0$, so $c = 4$. Then $x^2 + 4x + 4 = (x+2)^2$, which is 0 only at $x = -2$.

(c) With $c = 3$ the bottom is $x^2 + 4x + 3 = (x+1)(x+3)$, which is 0 at -1 and -3. So r is discontinuous at both. For $x \neq -1$ the factor $x + 1$ cancels and $r(x) = \dfrac{1}{x+3}$, so $\lim_{x \to -1} r(x) = \dfrac{1}{2}$: Condition 2 holds. But canceling describes r only for $x \neq -1$; the formula for r still gives $\dfrac{0}{0}$ at -1, so $r(-1)$ is not defined and Condition 1 fails.

6. Decide whether each statement is true for all functions f and g and every number a. Prove a true one with the results of this lesson, and give a counterexample to a false one, checking the conditions.

(a) If g is continuous at a and f is continuous at a, then $f\big(g(x)\big)$ is continuous at a.

(b) If f is continuous at a and $f(a) > 0$, then $\sqrt{f(x)}$ is continuous at a.

ANSWER

(a) False: for example $a = 0$, $g(x) = x + 1$, and $f(u) = 0$ for $u < 1$, $f(u) = 1$ for $u \geq 1$. (b) True.

SOLUTION

(a) False. Take $a = 0$, $g(x) = x + 1$, and $f(u) = 0$ for $u < 1$, $f(u) = 1$ for $u \geq 1$. The polynomial g is continuous at 0. For u near 0 (all $u < 1$), $f(u) = 0$, so $\lim_{u \to 0} f(u) = 0 = f(0)$ and f is continuous at 0. But $f\big(g(x)\big) = 0$ for $x < 0$, where $x + 1 < 1$, and $f\big(g(x)\big) = 1$ for $x \geq 0$. The one-sided limits at 0 are 0 and 1, so the limit does not exist: Condition 2 fails, and $f \circ g$ is not continuous at 0. The composition result needs f continuous at the inner value $g(0) = 1$, not at 0, and f jumps at 1.

(b) True. $\sqrt{f(x)}$ is the composition of the inner function f with the outer function $\sqrt{u}$. The inner function is continuous at a, and $\sqrt{u}$ is continuous at $u = f(a)$ because $f(a) > 0$ is inside the domain and not its end. By the composition result, $\sqrt{f(x)}$ is continuous at a.

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