Calculus 1 Unit 1 worksheet: Making a piecewise function continuous (answer key)

ANSWERS AT A GLANCE

1. (a) $k = -1$ (b) $k = \frac{3}{4}$ (c) $k = -6$ (d) $k = 4$
2. (a) At 1 both sides arrive at height 0 whatever a is. (b) $a = 15$ and $b = 25$. (c) 60 centimeters, at $x = 3$.
3. $k = -4$ or $k = 2$. The pieces meet at height 16 when $k = -4$, and at height 4 when $k = 2$.
4. (a) $k = 1$. No: f is not continuous at -1, where $f(-1)$ is not defined. It is continuous on $(-\infty, -1)$ and on $(-1, \infty)$. (b) $k = -6$, and then f is continuous at every number.
5. $c = -\frac{3}{2}$ and $k = -\frac{5}{4}$
6. (a) $k = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}$ or $\frac{17\pi}{18}$: four values. (b) $-5 \le c \le -1$.

WORKED SOLUTIONS

1. Find the constant k that makes each function continuous on its whole domain.

(a) $f(x) = \begin{cases} 3x + k, & x < 2 \\ x^2 - k, & x \ge 2 \end{cases}$

(b) $g(x) = \begin{cases} kx^2, & x \le 3 \\ 2x + k, & x > 3 \end{cases}$

(c) $p(x) = \begin{cases} \frac{k}{x}, & x < -2 \\ x + 5, & x \ge -2 \end{cases}$

(d) $q(x) = \begin{cases} k\sqrt{x}, & 0 \le x < 4 \\ x^2 - 2k, & x \ge 4 \end{cases}$

ANSWER

(a) $k = -1$ (b) $k = \frac{3}{4}$ (c) $k = -6$ (d) $k = 4$

SOLUTION

In each function both rules are continuous on their own open intervals (polynomials, $\frac{k}{x}$ for $x < -2$ where $x \ne 0$, and $k\sqrt{x}$ for $0 < x < 4$), so only the boundary needs an equation: left limit equals the value equals the right limit.

(a) At 2: the left limit is $3(2) + k = 6 + k$ and $f(2) = 4 - k$. So $6 + k = 4 - k$, $2k = -2$, $k = -1$. Both sides arrive at 5.

(b) At 3: $g(3) = 9k$, also the left limit; the right limit is $2(3) + k = 6 + k$. So $9k = 6 + k$, $8k = 6$, $k = \frac{3}{4}$. Both sides arrive at $\frac{27}{4}$.

(c) At -2: the left limit is $\frac{k}{-2}$ and $p(-2) = -2 + 5 = 3$. So $-\frac{k}{2} = 3$, $k = -6$. Check: $\frac{-6}{-2} = 3$.

(d) At 4: the left limit is $k\sqrt{4} = 2k$ and $q(4) = 16 - 2k$. So $2k = 16 - 2k$, $k = 4$. Both sides arrive at 8. At 0, the end of the domain, q is continuous from the right since $4\sqrt{x} \to 0 = q(0)$.

2. Layla designs a skateboard ramp. Its height is $h(x)$ centimeters at a horizontal distance of x meters from a mark on the ground:

$$h(x) = \begin{cases} 0, & 0 \le x < 1 \\ a(x-1)^2, & 1 \le x \le 3 \\ bx - a, & 3 < x \le 5 \end{cases}$$

The ramp starts on flat ground, curves up, then rises in a straight line to its lip at $x = 5$, which is 110 centimeters high. The surface may turn a corner, but it must never have a step.

(a) Explain why the boundary at $x = 1$ gives no equation for a. (b) Find a and b. (c) How high is the ramp where the curve meets the straight part?

Free sample from math.wiki for classroom use. Do not post. math.wiki

ANSWER

(a) At 1 both sides arrive at height 0 whatever a is. (b) $a = 15$ and $b = 25$. (c) 60 centimeters, at $x = 3$.

SOLUTION

Each rule is a polynomial, so the height can only jump at the boundaries 1 and 3.

(a) The middle rule includes 1, so $h(1) = a(1-1)^2 = 0$. The left limit comes from the flat rule and is 0. Both are 0 for every a, so h is continuous at 1 and this boundary says nothing about a.

(b) **At 3.** The middle rule includes 3, so $h(3) = a(3-1)^2 = 4a$, also the left limit. The right limit is $3b - a$. No step means $4a = 3b - a$, that is $3b = 5a$.

The lip. $h(5) = 5b - a = 110$.

With $b = \dfrac{5a}{3}$: $\dfrac{25a}{3} - a = \dfrac{22a}{3} = 110$, so $a = 15$ and $b = 25$.

(c) $h(3) = 4(15) = 60$ centimeters, and the straight part gives 25(3) − 15 = 60 too. Check the lip: 25(5) − 15 = 110. The ramp turns a corner at 3, but continuity asks only that the heights meet.

3. Here the unknown constant is where the boundary sits. Let

$$f(x) = \begin{cases} x^2, & x < k \\ -2x + 8, & x \geq k \end{cases}$$

Find every boundary k that makes f continuous. For each one, give the height at which the two pieces meet.

ANSWER

$k = -4$ or $k = 2$. The pieces meet at height 16 when $k = -4$, and at height 4 when $k = 2$.

SOLUTION

Both rules are polynomials, continuous everywhere, so wherever the boundary is, only the boundary $x = k$ itself matters.

The left limit at k is k^2. The second rule includes k, so $f(k) = -2k + 8$, which is also the right limit. Continuity says

$$k^2 = -2k + 8, \qquad k^2 + 2k - 8 = 0, \qquad (k+4)(k-2) = 0$$

so $k = -4$ or $k = 2$: the boundary must sit where the parabola $y = x^2$ meets the line. Check by the heights: with $k = -4$ the left side arrives at $(-4)^2 = 16$ and the line gives −2 · (−4) + 8 = 16; with $k = 2$ both give 4. Two different continuous functions.

4. Let $f(x) = \dfrac{12}{x + k}$ for $x < 2$ and $f(x) = x^2$ for $x \geq 2$.

(a) Find the k that makes f continuous at 2. With that k, is f continuous at every number? If not, say where it fails and give the largest intervals on which f is continuous.

(b) Answer the same questions when the second rule is $x^2 - 7$ instead.

ANSWER

(a) $k = 1$. No: f is not continuous at −1, where $f(-1)$ is not defined. It is continuous on $(-\infty, -1)$ and on $(-1, \infty)$. (b) $k = -6$, and then f is continuous at every number.

SOLUTION

(a) The right rule includes 2, so $f(2) = 4$, also the right limit. If $2 + k \neq 0$, the left limit is $\dfrac{12}{2 + k}$ (if $2 + k = 0$ the left side runs off to infinity and nothing works). Continuity at 2 says $\dfrac{12}{2 + k} = 4$, so $2 + k = 3$

Free sample from math.wiki for classroom use. Do not post. math.wiki

and $k = 1$.

But then the left rule is $\dfrac{12}{x+1}$, whose bottom is 0 at $x = -1$, and $-1 < 2$ lies inside that piece. So $f(-1)$ is not defined and f is not continuous at -1 (the outputs run off to infinity there). The constant that joins the pieces at 2 breaks the left piece from the inside. Away from -1 every condition holds, so f is continuous on $(-\infty, -1)$ and on $(-1, \infty)$.

(b) Now $f(2) = 4 - 7 = -3$. Continuity at 2 says $\dfrac{12}{2+k} = -3$, so $2 + k = -4$ and $k = -6$. The left rule is $\dfrac{12}{x-6}$, whose bottom is 0 only at 6, which is not in $x < 2$. So the left piece is continuous on $(-\infty, 2)$, the right piece on $(2, \infty)$, and the pieces join at 2: f is continuous at every number.

5. Find the constants c and k that make f continuous at 2, where

$$f(x) = \begin{cases} \dfrac{\frac{6}{x} - 3}{x - 2}, & 0 < x < 2 \\ c, & x = 2 \\ kx + 1, & x > 2 \end{cases}$$

ANSWER

$c = -\dfrac{3}{2}$ and $k = -\dfrac{5}{4}$

SOLUTION

At 2 the left rule reads $\dfrac{3-3}{0} = \dfrac{0}{0}$, so its limit must be computed. Combine the fractions in the top: for $0 < x < 2$,

$$\frac{\frac{6}{x} - 3}{x - 2} = \frac{6 - 3x}{x(x-2)} = \frac{-3(x-2)}{x(x-2)} = -\frac{3}{x}$$

which tends to $-\dfrac{3}{2}$ as $x \to 2^-$. The right limit is $2k + 1$. The value c is on its own line, so two equations are needed:

$$-\frac{3}{2} = c = 2k + 1, \qquad c = -\frac{3}{2}, \quad 2k = -\frac{5}{2}, \quad k = -\frac{5}{4}$$

Inside the pieces nothing breaks: $-\dfrac{3}{x}$ is continuous for $0 < x < 2$, and $kx + 1$ is a line.

6. Let $f(x) = 2\sin(kx)$ for $x < 3$ and $f(x) = x + c$ for $x \geq 3$, where $k > 0$ and c are constants (sine of an angle in radians).

(a) Take $c = -2$. Find every k with $0 < k < \pi$ that makes f continuous.

(b) Find every c for which at least one $k > 0$ makes f continuous.

ANSWER

(a) $k = \dfrac{\pi}{18}, \dfrac{5\pi}{18}, \dfrac{13\pi}{18}$ or $\dfrac{17\pi}{18}$: four values. (b) $-5 \leq c \leq -1$.

SOLUTION

Both rules are continuous for every k and c, so only $x = 3$ matters. The left limit is $2\sin(3k)$ and $f(3) = 3 + c$, so continuity asks $2\sin(3k) = 3 + c$.

Free sample from math.wiki for classroom use. Do not post. math.wiki

(a) With $c = -2$, $f(3) = 1$, so $\sin(3k) = \dfrac{1}{2}$. With $0 < k < \pi$, the angle $3k$ runs over $0 < 3k < 3\pi$. In that range $\sin\theta = \dfrac{1}{2}$ at

$$\theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad \frac{\pi}{6} + 2\pi = \frac{13\pi}{6}, \quad \frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$$

and the next one, $\dfrac{25\pi}{6}$, is past $3\pi = \dfrac{18\pi}{6}$. Dividing by 3: $k = \dfrac{\pi}{18}, \dfrac{5\pi}{18}, \dfrac{13\pi}{18}, \dfrac{17\pi}{18}$. Check $k = \dfrac{13\pi}{18}$: $2\sin\dfrac{13\pi}{6} = 2 \cdot \dfrac{1}{2} = 1$.

(b) The left side arrives at $2\sin(3k)$, which is never below -2 or above 2. Every height y from -2 to 2 is reached: $s = \dfrac{y}{2}$ lies in $[-1, 1]$, the angle $\theta = \pi - \arcsin s$ lies in $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$ with $\sin\theta = s$, and $k = \dfrac{\theta}{3}$ is positive. So some $k > 0$ works exactly when $-2 \le 3 + c \le 2$, that is $-5 \le c \le -1$. The ends work: $c = -1$ with $k = \dfrac{\pi}{6}$ ($2\sin\dfrac{\pi}{2} = 2$), and $c = -5$ with $k = \dfrac{\pi}{2}$ ($2\sin\dfrac{3\pi}{2} = -2$). For any other c the value $3 + c$ is out of reach and f has a jump at 3 for every k.

Free sample from math.wiki for classroom use. Do not post. math.wiki