Calculus 1 Unit 1 worksheet: Continuity on an interval (answer key)

ANSWERS AT A GLANCE

1. (a) Continuous from the left, not from the right. (b) Continuous from the right, not from the left. (c) Neither.
2. (a) $(-\infty, 4]$, continuous from the left at 4 (b) $(-\infty, 0)$ and $(4, \infty)$ (c) $[-3, 1)$, continuous from the right at -3, and $(1, \infty)$ (d) $(0, 1)$ and $(1, \infty)$
3. (a) Yes. (b) No. (c) Yes. (d) Yes.
4. (a) $g(2) = 1$, $\lim_{x \to 2^-} g(x) = 1$, $\lim_{x \to 2^+} g(x) = 0$: continuous from the left, not from the right. (b) Only on $(-1, 0]$. (c) Only $(n, n+1]$, for every integer n.
5. (a) Yes. (b) No: $\lim_{x \to 2^+} k(x) = -3 \neq -2 = k(2)$. (c) $[-4, 2]$ and $(2, 6]$.
6. (a) $\cos x$ is continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with values in $[0, 1]$, inside $[0, \infty)$ where $\sqrt{u}$ is continuous. (b) No: $\cos x < 0$ on $\left(\frac{\pi}{2}, \pi\right]$, so s is not defined there. (c) $\frac{\sqrt{2}}{2}$

WORKED SOLUTIONS

1. For each function, decide whether it is continuous from the left at the given number, and whether it is continuous from the right. Give the one-sided limits and the value that decide each answer. ($\lfloor x \rfloor$ is the floor function.)

(a) $f(x) = \sqrt[4]{8 - 2x}$ at 4

(b) $p(x) = \sqrt{x}\,\lfloor x \rfloor$ at 4

(c) $r(x) = x^2$ for $x \neq 3$, $r(3) = 1$, at 3

ANSWER

(a) Continuous from the left, not from the right. (b) Continuous from the right, not from the left. (c) Neither.

SOLUTION

(a) The fourth root needs $8 - 2x \geq 0$, so $x \leq 4$, and $f(4) = \sqrt[4]{0} = 0$. As $x \to 4^-$ the inner values $8 - 2x$ approach 0 through $[0, \infty)$, where the fourth root is continuous, so $\lim_{x \to 4^-} f(x) = 0 = f(4)$: continuous from the left. For $x > 4$, f is not defined, so $\lim_{x \to 4^+} f(x)$ does not exist: not continuous from the right.

(b) $p(4) = \sqrt{4} \cdot 4 = 8$. For $4 < x < 5$, $\lfloor x \rfloor = 4$, so $\lim_{x \to 4^+} p(x) = 2 \cdot 4 = 8 = p(4)$: continuous from the right. For $3 < x < 4$, $\lfloor x \rfloor = 3$, so $\lim_{x \to 4^-} p(x) = 2 \cdot 3 = 6 \neq 8$: not continuous from the left.

(c) Near 3, $r(x) = x^2$, so both one-sided limits are 9. The value is $r(3) = 1 \neq 9$, so r is continuous from neither side.

2. Find the largest intervals on which each function is continuous. At each end an interval includes, say which one-sided condition holds there.

(a) $\sqrt{12 - 3x}$

(b) $\ln(x^2 - 4x)$

(c) $\sqrt{x + 3} - \dfrac{1}{x - 1}$

(d) $\dfrac{\sqrt{x}}{\ln x}$

ANSWER

(a) $(-\infty, 4]$, continuous from the left at 4 (b) $(-\infty, 0)$ and $(4, \infty)$ (c) $[-3, 1)$, continuous from the right at -3, and $(1, \infty)$ (d) $(0, 1)$ and $(1, \infty)$

SOLUTION

(a) The square root needs $12 - 3x \geq 0$, so $x \leq 4$. It is the square root of a polynomial whose values on $(-\infty, 4]$ lie in $[0, \infty)$, so by the composition result it is continuous on $(-\infty, 4]$; at the end 4 this means continuous from the left, with $\lim_{x \to 4^-} \sqrt{12 - 3x} = 0$, the value there.

(b) $\ln u$ is continuous on $(0, \infty)$, and $x^2 - 4x = x(x - 4) > 0$ exactly when $x < 0$ or $x > 4$. So the intervals are $(-\infty, 0)$ and $(4, \infty)$. Neither 0 nor 4 can be added, since $\ln 0$ is not defined.

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(c) $\sqrt{x+3}$ is continuous on $[-3, \infty)$, and $\dfrac{1}{x-1}$ on every interval avoiding 1. By the sum result the function is continuous on $[-3, 1)$ and $(1, \infty)$. At -3 both terms are continuous from the right (the second in the two-sided sense), so the difference is continuous from the right there, with value $0 - \dfrac{1}{-4} = \dfrac{1}{4}$.

(d) $\ln x$ needs $x > 0$, so 0 is not in the domain even though $\sqrt{x}$ is defined there. The bottom $\ln x$ is 0 at $x = 1$. By the quotient result: $(0, 1)$ and $(1, \infty)$.

3. Let $f(x) = \dfrac{\sqrt{9 - x^2}}{x + 1}$. Is f continuous on each interval? Give a reason, and check any one-sided condition the interval needs.

(a) $[0, 3]$ (b) $[-3, 0]$ (c) $[-3, -1)$ (d) $(-1, 3]$

ANSWER

(a) Yes. (b) No. (c) Yes. (d) Yes.

SOLUTION

The top needs $9 - x^2 \geq 0$, so $-3 \leq x \leq 3$, and on $[-3, 3]$ it is the square root of a polynomial with values in $[0, \infty)$, so it is continuous on $[-3, 3]$. The bottom $x + 1$ is 0 only at -1. By the quotient result, f is continuous on every interval inside $[-3, 3]$ that avoids -1, with one-sided continuity at -3 and 3.

(a) Yes. $[0, 3]$ avoids -1. At 3: $\lim_{x \to 3^-} f(x) = \dfrac{\sqrt{0}}{4} = 0 = f(3)$, so f is continuous from the left at 3.

(b) No. $f(-1)$ would need a division by 0, and -1 lies inside $[-3, 0]$.

(c) Yes. Inside, f is continuous; at -3, $\lim_{x \to -3^+} f(x) = \dfrac{\sqrt{0}}{-2} = 0 = f(-3)$, so f is continuous from the right. The end -1 is not included, so nothing is asked there.

(d) Yes. The open end -1 is not included, and at 3 the check in (a) holds.

4. Let $g(x) = \lfloor 3 - x \rfloor$, the floor of $3 - x$.

(a) Find $g(2)$, $\lim_{x \to 2^-} g(x)$ and $\lim_{x \to 2^+} g(x)$. Is g continuous from the left at 2? From the right?

(b) On which of $[-1, 0)$, $(-1, 0]$ and $(-1, 1)$ is g continuous?

(c) For an integer n, which of $[n, n + 1)$, $(n, n + 1]$ and $[n, n + 1]$ is g continuous on? Explain.

ANSWER

(a) $g(2) = 1$, $\lim_{x \to 2^-} g(x) = 1$, $\lim_{x \to 2^+} g(x) = 0$: continuous from the left, not from the right. (b) Only on $(-1, 0]$. (c) Only $(n, n + 1]$, for every integer n.

SOLUTION

(a) $g(2) = \lfloor 1 \rfloor = 1$. For x a little less than 2, $3 - x$ is a little more than 1, so $g(x) = 1$ and $\lim_{x \to 2^-} g(x) = 1 = g(2)$: continuous from the left. For x a little more than 2, $3 - x$ is a little less than 1, so $g(x) = 0$ and $\lim_{x \to 2^+} g(x) = 0 \neq 1$: not continuous from the right.

(b) For $-1 < x < 0$, $3 < 3 - x < 4$, so $g(x) = 3$, a constant: g is continuous on $(-1, 0)$. - $[-1, 0)$ needs continuity from the right at -1. But $g(-1) = \lfloor 4 \rfloor = 4$ while $\lim_{x \to -1^+} g(x) = 3$. Not continuous. - $(-1, 0]$ needs continuity from the left at 0: $g(0) = \lfloor 3 \rfloor = 3 = \lim_{x \to 0^-} g(x)$. Continuous. - $(-1, 1)$ has 0 inside it, where both sides are needed. For x a little more than 0, $g(x) = 2$, so $\lim_{x \to 0^+} g(x) = 2 \neq 3 = g(0)$. Not continuous.

(c) The same reasoning works at every integer n: g is the constant $2 - n$ on $(n, n + 1)$, $g(n) = 3 - n$, the limit from the left at n is $3 - n$ and the limit from the right is $2 - n$. So g is continuous from the left and not from the right at every integer, the mirror image of $\lfloor x \rfloor$. The right end $n + 1$ of $(n, n + 1]$ needs only

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continuity from the left, which holds, so g is continuous on $(n, n+1]$. The intervals $[n, n+1)$ and $[n, n+1]$ include the left end n, which needs continuity from the right, which fails.

5. Let

$$k(x) = \begin{cases} \sqrt{x+4} & -4 \le x < 0 \\ 2 - x^2 & 0 \le x \le 2 \\ x - 5 & 2 < x \le 6 \end{cases}$$

(a) Is k continuous on $[-4, 2]$? Check every condition the definition asks for.

(b) Is k continuous on $[2, 6]$?

(c) Find the largest intervals inside $[-4, 6]$ on which k is continuous.

ANSWER

(a) Yes. (b) No: $\lim_{x \to 2^+} k(x) = -3 \neq -2 = k(2)$. (c) $[-4, 2]$ and $(2, 6]$.

SOLUTION

Each piece is continuous on its own interval: $\sqrt{x+4}$ on $[-4, \infty)$, and the two polynomials everywhere. The questions are at −4, 0, 2 and 6.

(a) At −4: $k(-4) = \sqrt{0} = 0$ and $\lim_{x \to -4^+} \sqrt{x+4} = 0$, so k is continuous from the right. At 0, which is inside $[-4, 2]$ and needs both sides: $\lim_{x \to 0^-} \sqrt{x+4} = 2$, $\lim_{x \to 0^+} (2 - x^2) = 2$ and $k(0) = 2$, so k is continuous at 0. At 2: $k(2) = 2 - 4 = -2$ and $\lim_{x \to 2^-} (2 - x^2) = -2$, so k is continuous from the left. Every other number of $(-4, 2)$ lies inside one piece. So k is continuous on $[-4, 2]$.

(b) The end 2 needs continuity from the right: $\lim_{x \to 2^+} (x - 5) = -3$, but $k(2) = -2$. So k is not continuous on $[2, 6]$.

(c) On $(2, 6]$, $k(x) = x - 5$, a polynomial, so k is continuous there (from the left at 6). No interval containing 2 together with numbers to its right works, by (b). So the largest intervals are $[-4, 2]$ and $(2, 6]$.

6. Let $s(x) = \sqrt{\cos x}$ (radians).

(a) Use the composition result to show that s is continuous on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

(b) Is s continuous on $[0, \pi]$? Explain.

(c) Use continuity to find $\lim_{x \to \pi/3} s(x)$.

ANSWER

(a) $\cos x$ is continuous on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ with values in $[0, 1]$, inside $[0, \infty)$ where $\sqrt{u}$ is continuous. (b) No: $\cos x < 0$ on $\left(\dfrac{\pi}{2}, \pi\right]$, so s is not defined there. (c) $\dfrac{\sqrt{2}}{2}$

SOLUTION

(a) The inner function $\cos x$ is continuous on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, and for x in that interval $0 \le \cos x \le 1$, so every inner value lies in $J = [0, \infty)$. The square root is continuous on $[0, \infty)$. By the composition result, s is continuous on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, including one-sided continuity at both ends, where $s = 0$.

(b) No. For $\dfrac{\pi}{2} < x \le \pi$, $\cos x < 0$, so $s(x)$ is not defined: continuity on $[0, \pi]$ fails at every number of $\left(\dfrac{\pi}{2}, \pi\right]$. ($s$ is continuous on $\left[0, \dfrac{\pi}{2}\right]$.)

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(c) $\frac{\pi}{3}$ lies inside $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, where s is continuous, so the limit is the value:

$$\lim_{x \to \pi/3} \sqrt{\cos x} = \sqrt{\cos \tfrac{\pi}{3}} = \sqrt{\tfrac{1}{2}} = \frac{\sqrt{2}}{2}$$

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