Calculus 1 Unit 1 worksheet: The epsilon delta definition of a limit (answer key)

ANSWERS AT A GLANCE

1. (a) $4.97 < x < 5.03$ (b) $\delta = 0.03$ (c) $\varepsilon = 0.24$
2. (a) True. (b) False; for example $f(x) = \dfrac{x}{2}$ at $a = 0$, where $x = 0.15$ fails.
3. (a) $\delta = \dfrac{\varepsilon}{2}$ works (proof in the key). (b) Canceling the factor $x - 2$, which needs $x \neq 2$.
4. (a) $\varepsilon = 1$ (the outputs run from 1 to $\sqrt[3]{15}$, about 2.466, ends left out) (b) for example $x = \dfrac{27}{8}$, whose output is $\dfrac{3}{2}$
5. $\delta = \min(1, \varepsilon)$ works (proof in the key); for $\varepsilon = 0.3$, $\delta = 0.3$.
6. (a) $f(x) = x + 2$ for $x > 2$, and $f(x) = -(x + 2)$ for $x < 2$. (b) With $\varepsilon = 2$ no δ works for any L, because two allowed inputs always have outputs exactly 8 apart; the limit does not exist.
7. (a) For example $x = -2.009$, whose output 10.099243 is more than 0.08 from 10. (b) $|3x - 5|$ is smallest, not largest, at $x = -1$; the correct bound is $|3x - 5| < 14$, and $\delta = \min\left(1, \dfrac{\varepsilon}{14}\right)$ works (proof in the key).

WORKED SOLUTIONS

1. Let $f(x) = -4x - 7$, so that $\lim_{x \to 5} f(x) = -27$.

(a) Find the interval of inputs x whose outputs satisfy $|f(x) + 27| < 0.12$.

(b) Find the largest δ that guarantees an output within 0.12 of -27.

(c) Suppose $\delta = 0.06$ is used instead. Find the smallest ε for which this δ works.

> **ANSWER**
> (a) $4.97 < x < 5.03$ (b) $\delta = 0.03$ (c) $\varepsilon = 0.24$
>
> **SOLUTION**
> Scratch work first: $f(x) + 27 = -4x + 20 = -4(x - 5)$, so the output distance is $|f(x) + 27| = 4\,|x - 5|$, exactly 4 times the input distance.
>
> (a) $4\,|x - 5| < 0.12$ exactly when $|x - 5| < \dfrac{0.12}{4} = 0.03$, that is, when $4.97 < x < 5.03$.
>
> (b) That interval is centered on 5 and reaches 0.03 to each side, so $\delta = 0.03$ works. A larger δ lets in inputs outside the interval, whose outputs miss the band, so $\delta = 0.03$ is the largest.
>
> (c) If $0 < |x - 5| < 0.06$, then $|f(x) + 27| = 4\,|x - 5| < 4 \cdot 0.06 = 0.24$, so this δ works for $\varepsilon = 0.24$. Inputs near the ends of the window give outputs as close to 0.24 from -27 as you like, so no smaller ε is guaranteed: the smallest is $\varepsilon = 0.24$.

2. In each statement, $\lim_{x \to a} f(x) = L$. Decide whether the statement is true for every such f, a and L. Give a reason, or a counterexample with numbers.

(a) If $\delta = 0.2$ works for $\varepsilon = 0.1$, then $\delta = 0.15$ works for $\varepsilon = 0.1$.

(b) If $\delta = 0.2$ works for $\varepsilon = 0.1$, then $\delta = 0.2$ works for $\varepsilon = 0.05$.

> **ANSWER**
> (a) True. (b) False; for example $f(x) = \dfrac{x}{2}$ at $a = 0$, where $x = 0.15$ fails.
>
> **SOLUTION**
> (a) True. Every x with $0 < |x - a| < 0.15$ also has $0 < |x - a| < 0.2$, so its output is within 0.1 of L. A narrower strip holds fewer inputs.
>
> (b) False. Take $f(x) = \dfrac{x}{2}$, $a = 0$, $L = 0$. If $0 < |x| < 0.2$, then $\left|\dfrac{x}{2}\right| < 0.1$, so $\delta = 0.2$ works for $\varepsilon = 0.1$. But $x = 0.15$ is allowed by $\delta = 0.2$ and $\left|\dfrac{0.15}{2}\right| = 0.075 \geq 0.05$. A smaller tolerance may need a tighter window.

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3. Let $f(x) = \dfrac{2x^2 - 3x - 2}{x - 2}$ for $x \neq 2$.

(a) Prove from the definition that $\lim_{x \to 2} f(x) = 5$.

(b) Which step of your proof uses the condition $0 < |x - 2|$, and not only $|x - 2| < \delta$?

ANSWER

(a) $\delta = \dfrac{\varepsilon}{2}$ works (proof in the key). (b) Canceling the factor $x - 2$, which needs $x \neq 2$.

SOLUTION

(a) Scratch work. The top factors: $2x^2 - 3x - 2 = (2x + 1)(x - 2)$. So for $x \neq 2$, $f(x) = 2x + 1$, and

$$|f(x) - 5| = |2x - 4| = 2|x - 2|.$$

The output distance is twice the input distance, so $\delta = \dfrac{\varepsilon}{2}$ should work.

Proof. Let $\varepsilon > 0$ and choose $\delta = \dfrac{\varepsilon}{2}$. If $0 < |x - 2| < \delta$, then $x \neq 2$, so $f(x) = 2x + 1$ and

$$|f(x) - 5| = 2|x - 2| < 2\delta = \varepsilon.$$

Since ε was any positive number, $\lim_{x \to 2} f(x) = 5$.

(b) The step $f(x) = 2x + 1$ cancels $x - 2$, which is allowed only for $x \neq 2$; the condition $0 < |x - 2|$ is exactly what guarantees $x \neq 2$. Without it the definition would also ask about $x = 2$, where f is not even defined.

4. Zoe uses $\delta = 7$ for the limit $\lim_{x \to 8} \sqrt[3]{x} = 2$. This item runs the definition backward: the window is fixed, and the question is which tolerances it meets.

(a) Find the smallest ε for which her δ works.

(b) Zoe notices that to the right of 8 every output is within 0.5 of 2, and says her δ also works for $\varepsilon = 0.5$. Find an allowed input that shows she is wrong.

ANSWER

(a) $\varepsilon = 1$ (the outputs run from 1 to $\sqrt[3]{15}$, about 2.466, ends left out) (b) for example $x = \dfrac{27}{8}$, whose output is $\dfrac{3}{2}$

SOLUTION

(a) The allowed inputs are $1 < x < 15$ with $x \neq 8$. The cube root is increasing, so the outputs run from $\sqrt[3]{1} = 1$ to $\sqrt[3]{15} \approx 2.466$, ends not included, and skip only $\sqrt[3]{8} = 2$.

On the left the outputs come as close as you like to 1, a distance of 2 − 1 = 1 from the limit, without reaching it. On the right they stay below $\sqrt[3]{15}$, at most $\sqrt[3]{15} - 2 \approx 0.466$ from the limit. So every allowed input has $|\sqrt[3]{x} - 2| < 1$, and $\delta = 7$ works for $\varepsilon = 1$. For any $\varepsilon < 1$, inputs just above 1 have outputs just above 1, more than ε below 2, so $\delta = 7$ fails. The smallest is $\varepsilon = 1$.

(b) Take $x = \dfrac{27}{8} = 3.375$: $\left|\dfrac{27}{8} - 8\right| = \dfrac{37}{8} < 7$, so it is allowed, but $\sqrt[3]{\frac{27}{8}} = \dfrac{3}{2}$ and $\left|\dfrac{3}{2} - 2\right| = \dfrac{1}{2}$, which is not less than 0.5. The left side decides, because the curve climbs faster to the left of 8 than to the right.

5. Prove that $\lim_{x \to 4} \dfrac{x}{x - 2} = 2$. Then find the δ your rule gives for $\varepsilon = 0.3$.

ANSWER

$\delta = \min(1, \varepsilon)$ works (proof in the key); for $\varepsilon = 0.3$, $\delta = 0.3$.

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SOLUTION

Scratch work. Combine over a common denominator:

$$\left|\frac{x}{x-2}-2\right| = \left|\frac{x-2(x-2)}{x-2}\right| = \frac{|4-x|}{|x-2|} = \frac{|x-4|}{|x-2|}.$$

The factor $\dfrac{1}{|x-2|}$ grows without bound near $x=2$, so keep the strip away from 2. Agree that $\delta \le 1$. Then $|x-4|<1$ gives $3<x<5$, so $x-2>1$ and $\dfrac{1}{|x-2|}<1$. Under that agreement the output distance is less than $|x-4|$, which is less than ε once $|x-4|<\varepsilon$.

Proof. Let $\varepsilon>0$ and choose $\delta=\min(1,\varepsilon)$. If $0<|x-4|<\delta$, then $|x-4|<1$, so $x>3$ and $\dfrac{1}{|x-2|}<1$; and $|x-4|<\varepsilon$. Therefore

$$\left|\frac{x}{x-2}-2\right| = \frac{|x-4|}{|x-2|} < |x-4| < \varepsilon.$$

So the limit is 2.

For $\varepsilon=0.3$ the rule gives $\delta=\min(1,0.3)=0.3$. An agreement such as $\delta\le 2$ would not do: the strip $2<x<6$ comes as close to 2 as you like, and no bound on $\dfrac{1}{|x-2|}$ holds there.

6. Let $f(x)=\dfrac{x^2-4}{|x-2|}$ for $x\neq 2$.

(a) Write $f(x)$ without absolute value bars for $x>2$ and for $x<2$.

(b) Use the definition to show that no number L is $\lim_{x\to 2} f(x)$.

ANSWER

(a) $f(x)=x+2$ for $x>2$, and $f(x)=-(x+2)$ for $x<2$. (b) With $\varepsilon=2$ no δ works for any L, because two allowed inputs always have outputs exactly 8 apart; the limit does not exist.

SOLUTION

(a) $x^2-4=(x-2)(x+2)$. For $x>2$, $|x-2|=x-2$, so $f(x)=x+2$. For $x<2$, $|x-2|=-(x-2)$, so $f(x)=-(x+2)$.

(b) Suppose some number L were the limit. Take $\varepsilon=2$. Then there would be a $\delta>0$ with $|f(x)-L|<2$ for every x with $0<|x-2|<\delta$. Let $t=\dfrac{\delta}{2}$. The inputs $2+t$ and $2-t$ are both allowed, and by (a)

$$f(2+t)=4+t, \qquad f(2-t)=-(4-t)=t-4.$$

By the triangle inequality,

$$\begin{aligned} 8=|(4+t)-(t-4)| &\le |(4+t)-L|+|L-(t-4)| \\ &< 2+2=4, \end{aligned}$$

which is impossible. So no δ answers $\varepsilon=2$, whatever L is, and $\lim_{x\to 2} f(x)$ does not exist. In the picture: the outputs sit near 4 just right of 2 and near -4 just left of it, and a band of height 4 cannot hold both.

7. Grace writes this scratch work for $\lim_{x\to -2}(3x^2+x)=10$: "$|3x^2+x-10| = |x+2|\,|3x-5|$. Agree that $\delta\le 1$, so $-3<x<-1$. At $x=-1$ the second factor is $|3(-1)-5|=8$, so $|3x-5|<8$, and $\delta=\min\left(1,\dfrac{\varepsilon}{8}\right)$ works."

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(a) For $\varepsilon = 0.08$ her rule gives $\delta = 0.01$. Find an allowed input whose output is not within 0.08 of 10.

(b) Find her mistake, and write a correct proof that $\lim_{x \to -2}(3x^2 + x) = 10$.

ANSWER

(a) For example $x = -2.009$, whose output 10.099243 is more than 0.08 from 10. (b) $|3x - 5|$ is smallest, not largest, at $x = -1$; the correct bound is $|3x - 5| < 14$, and $\delta = \min\left(1, \dfrac{\varepsilon}{14}\right)$ works (proof in the key).

SOLUTION

(a) Take $x = -2.009$. Then $0 < |x + 2| = 0.009 < 0.01$, so x is allowed, but

$$3(-2.009)^2 + (-2.009) = 12.108243 - 2.009 = 10.099243,$$

which is 0.099243 from 10, more than 0.08. (Any allowed input with $|x + 2|$ between about 0.0073 and 0.01 works, on either side of −2.)

(b) The mistake. On $-3 < x < -1$ the factor $3x - 5$ runs from −14 to −8, so $|3x - 5|$ runs from 8 up to 14. A bound must hold for every allowed x, so it must use the largest value, and Grace used the smallest. Already at $x = -2$ the factor is $|3(-2) - 5| = 11 > 8$, so near −2 the output distance is about $11|x + 2|$, more than her $8|x + 2|$.

Scratch work. $|3x^2 + x - 10| = |x + 2|\,|3x - 5|$. Agree that $\delta \le 1$. Then $|x + 2| < 1$ gives $-3 < x < -1$, so $-9 < 3x < -3$ and $-14 < 3x - 5 < -8$, so $|3x - 5| < 14$. Under that agreement $|3x^2 + x - 10| < 14|x + 2|$, which is less than ε once $|x + 2| < \dfrac{\varepsilon}{14}$.

Proof. Let $\varepsilon > 0$ and choose $\delta = \min\left(1, \dfrac{\varepsilon}{14}\right)$. If $0 < |x + 2| < \delta$, then $|x + 2| < 1$, so $|3x - 5| < 14$; and $|x + 2| < \dfrac{\varepsilon}{14}$. Therefore

$$|3x^2 + x - 10| = |x + 2|\,|3x - 5| < \frac{\varepsilon}{14} \cdot 14 = \varepsilon.$$

Since ε was any positive number, $\lim_{x \to -2}(3x^2 + x) = 10$.

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