Calculus 1 Unit 1 worksheet: Comparing growth rates at infinity (answer key)

ANSWERS AT A GLANCE

1. (a) ∞ (b) 0 (c) 0 (d) $\frac{5}{2}$
2. 2
3. $(\ln x)^{50}$, $100x^2$, $x^2 \ln x$, $x^{2.1}$, $\frac{x^3}{\ln x}$, $\frac{2^x}{x^{10}}$
4. (a) 0 (b) 0
5. (a) ∞; B grows faster. (b) $A(30) = 810{,}000$ views and $B(30) \approx 87$ views; the limit only describes the long run. (c) 0
6. (a) False. (b) True. (c) False: they grow at the same rate.

WORKED SOLUTIONS

1. Find each limit with the ladder, writing ∞ for a quotient that grows without bound. Name the rung comparison you use.

(a) $\lim_{x \to \infty} \frac{x^{3/2}}{(\ln x)^6}$

(b) $\lim_{x \to \infty} \frac{\ln(x^3)}{\sqrt[4]{x}}$

(c) $\lim_{x \to \infty} \frac{x^8 \, 2^x}{3^x}$

(d) $\lim_{x \to \infty} \frac{\ln(x^5)}{\ln(x^2)}$

ANSWER

(a) ∞ (b) 0 (c) 0 (d) $\frac{5}{2}$

SOLUTION

(a) By the ladder with $q = 6$ and $p = \frac{3}{2}$, $\frac{(\ln x)^6}{x^{3/2}} \to 0$. The quotient asked for is its positive reciprocal, so it grows without bound: the limit is ∞.

(b) $\ln(x^3) = 3 \ln x$, so the quotient is $3 \cdot \frac{\ln x}{x^{1/4}}$. By the ladder with $q = 1$ and $p = \frac{1}{4}$ this goes to $3 \cdot 0 = 0$.

(c) $\frac{2^x}{3^x} = \left(\frac{2}{3}\right)^x = \frac{1}{1.5^x}$, so the quotient is $\frac{x^8}{1.5^x}$, a power over an exponential with base $1.5 > 1$. By the ladder the limit is 0.

(d) For $x > 1$, $\frac{\ln(x^5)}{\ln(x^2)} = \frac{5 \ln x}{2 \ln x} = \frac{5}{2}$, a constant. The limit is $\frac{5}{2}$, so the two logarithms grow at the same rate.

2. Find the limit. Decide first which term on the top and which on the bottom grows fastest.

$$\lim_{x \to \infty} \frac{6x^4 \ln x + x^4}{3x^4 \ln x + 4x^4}$$

ANSWER

2

SOLUTION

Each line has a term $x^4 \ln x$ and a term that is a multiple of x^4. Since $\frac{x^4 \ln x}{x^4} = \ln x \to \infty$, the log factor puts $x^4 \ln x$ on a higher rung, so it is the fastest term on top and on the bottom. Divide top and bottom by $x^4 \ln x$:

$$\frac{6 + \frac{1}{\ln x}}{3 + \frac{4}{\ln x}}.$$

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Since $\ln x \to \infty$, both fractions with $\ln x$ below go to 0. The bottom goes to $3 \neq 0$, so by the quotient law the limit is 2. (Treating x^4 and $x^4 \ln x$ as the same rung, and getting $\dfrac{6+1}{3+4}$, ignores the log factor.)

3. Rank these six functions from slowest growing to fastest. Justify each neighboring pair in your ranking with the limit of a quotient.

$$x^{2.1}, \quad \frac{2^x}{x^{10}}, \quad 100x^2, \quad (\ln x)^{50}, \quad \frac{x^3}{\ln x}, \quad x^2 \ln x$$

ANSWER

$(\ln x)^{50}$, $100x^2$, $x^2 \ln x$, $x^{2.1}$, $\dfrac{x^3}{\ln x}$, $\dfrac{2^x}{x^{10}}$

SOLUTION

- $\dfrac{(\ln x)^{50}}{100x^2} = \dfrac{1}{100} \cdot \dfrac{(\ln x)^{50}}{x^2} \to 0$ by the ladder with $q = 50$, $p = 2$: the large exponent 50 does not lift a logarithm past a power.
- $\dfrac{x^2 \ln x}{100x^2} = \dfrac{\ln x}{100} \to \infty$: the factor 100 cannot keep $100x^2$ above $x^2 \ln x$.
- $\dfrac{x^2 \ln x}{x^{2.1}} = \dfrac{\ln x}{x^{0.1}} \to 0$, so $x^{2.1}$ is above $x^2 \ln x$.
- $\dfrac{x^{2.1}}{x^3 / \ln x} = \dfrac{\ln x}{x^{0.9}} \to 0$, so $\dfrac{x^3}{\ln x}$ is above $x^{2.1}$.
- $\dfrac{x^3 / \ln x}{2^x / x^{10}} = \dfrac{x^{13}}{2^x} \cdot \dfrac{1}{\ln x} \to 0 \cdot 0 = 0$ (the second factor goes to 0 because $\ln x \to \infty$), so $\dfrac{2^x}{x^{10}}$ is the fastest.

By the transitive rule the chain holds: $(\ln x)^{50}$, $100x^2$, $x^2 \ln x$, $x^{2.1}$, $\dfrac{x^3}{\ln x}$, $\dfrac{2^x}{x^{10}}$.

4. Find each limit. Use a substitution that turns it into a quotient from the ladder.

(a) $\displaystyle\lim_{x \to \infty} \frac{\ln(\ln x)}{\ln x}$ (b) $\displaystyle\lim_{x \to \infty} x^6 e^{-x^2}$

ANSWER

(a) 0 (b) 0

SOLUTION

(a) Put $t = \ln x$. Then $t \to \infty$ exactly when $x \to \infty$, and the quotient is $\dfrac{\ln t}{t}$, which goes to 0 by the ladder with $q = p = 1$. The limit is 0: $\ln x$ grows faster than $\ln(\ln x)$.

(b) The shape is $\infty \cdot 0$. For $x > 0$ put $t = x^2$, so $t \to \infty$ exactly when $x \to \infty$, $x^6 = t^3$ and $e^{-x^2} = \dfrac{1}{e^t}$. Then

$$x^6 e^{-x^2} = \frac{t^3}{e^t},$$

which goes to 0 by the ladder with $p = 3$ and $a = e > 1$. The limit is 0.

5. Layla tracks two videos she posted online. Her made-up models give the total views t days after posting as $A(t) = 900t^2$ for the first video and $B(t) = 5 \cdot 1.1^t$ for the second.

(a) Find $\lim_{t \to \infty} \dfrac{B(t)}{A(t)}$. Which model grows faster?

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(b) Find $A(30)$ and $B(30)$, to the nearest whole view. Explain why this does not contradict part (a).

(c) If both models held forever, what fraction of the two videos' combined views would belong to the first video in the long run? That is, find $\lim_{t\to\infty} \dfrac{A(t)}{A(t)+B(t)}$.

ANSWER

(a) ∞; B grows faster. (b) $A(30) = 810{,}000$ views and $B(30) \approx 87$ views; the limit only describes the long run. (c) 0

SOLUTION

(a) $\dfrac{B(t)}{A(t)} = \dfrac{5}{900} \cdot \dfrac{1.1^t}{t^2}$. By the ladder with $p = 2$ and $a = 1.1 > 1$, $\dfrac{t^2}{1.1^t} \to 0$, so its positive reciprocal grows without bound and the limit is ∞. The second video's model grows faster; the constant $\dfrac{5}{900}$ does not change the ranking.

(b) $A(30) = 900 \cdot 900 = 810{,}000$ views, and $B(30) = 5 \cdot 1.1^{30} \approx 5 \cdot 17.45 \approx 87$ views. The first video is far ahead at 30 days. The limit in (a) says only what happens past every point: it says the second model eventually leads for good, not when.

(c) Divide top and bottom by $A(t)$:

$$\frac{A(t)}{A(t)+B(t)} = \frac{1}{1+\dfrac{B(t)}{A(t)}}.$$

The bottom grows without bound by (a), so the fraction goes to 0. In the long run the first video's share of the combined views shrinks toward nothing.

6. Samir makes three claims. For each, say whether it is true or false. Support a true claim with a limit of a quotient, and a false one with a counterexample or the correct limit.

(a) If f and g grow at the same rate, then $\lim_{x\to\infty} \big(f(x) - g(x)\big) = 0$.

(b) $3^{x/2}$ grows slower than 2^x.

(c) $\ln(x^{100})$ grows faster than $\ln x$.

ANSWER

(a) False. (b) True. (c) False: they grow at the same rate.

SOLUTION

(a) False. Take $f(x) = x + \sqrt{x}$ and $g(x) = x$. Then $\dfrac{f(x)}{g(x)} = 1 + \dfrac{1}{\sqrt{x}} \to 1$, so they grow at the same rate, yet $f(x) - g(x) = \sqrt{x}$ grows without bound. Growth rates compare quotients, not differences.

(b) True. $3^{x/2} = \left(3^{1/2}\right)^x = (\sqrt{3})^x$, so

$$\frac{3^{x/2}}{2^x} = \left(\frac{\sqrt{3}}{2}\right)^x \to 0,$$

because $0 < \dfrac{\sqrt{3}}{2} < 1$ (it is about 0.866). Halving the exponent changes the base to $\sqrt{3}$, a lower rung than 2.

(c) False. $\ln(x^{100}) = 100 \ln x$, so $\dfrac{\ln(x^{100})}{\ln x} = 100$ for $x > 1$, and the limit is 100, a number between 0 and ∞. The two grow at the same rate; a constant factor never changes the rate.

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