Calculus 1 Unit 1 worksheet: Limits at infinity and horizontal asymptotes (answer key)

ANSWERS AT A GLANCE

1. (a) -4 (b) ∞ (the limit does not exist as a number) (c) $\frac{1}{2}$ (d) $\frac{9}{2}$
2. (b) No: at $x = -8{,}500$ the distance is $\frac{9}{8{,}501} > 0.001$. $N = -8{,}999$ works; in general $N = 1 - \frac{9}{t}$. (c) From below.
3. $y = \frac{9}{2}$ as $x \to \infty$ and $y = -\frac{3}{2}$ as $x \to -\infty$.
4. (a) 0 (b) $\frac{9}{2}$
5. (a) $y = x + 3$ (b) $y = x + 3$, the same line.
6. One example: $f(x) = -1 + \frac{2x}{\sqrt{x^2+1}}$. Other answers are possible; any correct one must be checked at both ends.

WORKED SOLUTIONS

1. Find each limit. Divide the top and the bottom by the highest power of x in the bottom, then use the basic limit and the limit laws. If a limit is not a number, write ∞ or $-\infty$ to say how the outputs grow without bound.

(a) $\lim_{x\to\infty} \frac{8x^2 + 3x}{5 - 2x^2}$

(b) $\lim_{x\to-\infty} \frac{1 - 4x^3}{x^2 + 6}$

(c) $\lim_{x\to\infty} \frac{\sqrt{x} + 3x^2}{6x^2 - x\sqrt{x}}$

(d) $\lim_{x\to\infty} \frac{(3x+1)^2(x-3)}{2x^3 + 1}$

ANSWER

(a) -4 (b) ∞ (the limit does not exist as a number) (c) $\frac{1}{2}$ (d) $\frac{9}{2}$

SOLUTION

(a) Divide by x^2: $\frac{8 + \frac{3}{x}}{\frac{5}{x^2} - 2}$. By the basic limit $\frac{3}{x} \to 0$ and $\frac{5}{x^2} \to 0$, so the top tends to 8 and the bottom to $-2 \neq 0$. The quotient law gives $\frac{8}{-2} = -4$.

(b) Divide by x^2: $\frac{\frac{1}{x^2} - 4x}{1 + \frac{6}{x^2}}$. The bottom tends to 1, but the top contains $-4x$, which is positive and grows without bound as $x \to -\infty$. So the quotient grows without bound: $\lim_{x\to-\infty} \frac{1 - 4x^3}{x^2 + 6} = \infty$. The limit does not exist as a number, and there is no horizontal asymptote on the left.

(c) Here $x\sqrt{x} = x^{3/2}$ and the highest power is x^2. Divide by x^2, using $\frac{x^{1/2}}{x^2} = \frac{1}{x^{3/2}}$ and $\frac{x^{3/2}}{x^2} = \frac{1}{x^{1/2}}$: $\frac{\frac{1}{x^{3/2}} + 3}{6 - \frac{1}{\sqrt{x}}}$. The basic limit holds for $p = \frac{3}{2}$ and $p = \frac{1}{2}$, so the limit is $\frac{3}{6} = \frac{1}{2}$.

(d) Divide by x^3, sharing it out as x^2 for the square and x for the last factor: $\frac{\left(3 + \frac{1}{x}\right)^2 \left(1 - \frac{3}{x}\right)}{2 + \frac{1}{x^3}}$. The power and product laws send the top to $3^2 \cdot 1 = 9$, the bottom tends to 2, and the limit is $\frac{9}{2}$.

2. Let $f(x) = \frac{2x + 7}{x - 1}$. This exercise proves $\lim_{x\to-\infty} f(x) = 2$ from the definition.

(a) Show that $|f(x) - 2| = \frac{9}{1 - x}$ for every $x < 1$.

(b) For the tolerance $t = 0.001$, is $N = -8{,}000$ far enough out, so that $|f(x) - 2| < 0.001$ for every $x < N$? If not, find an N that works. Then find an N, in terms of t, that works for every $t > 0$.

(c) On the far left, does the graph of f approach the line $y = 2$ from above or from below? Explain.

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ANSWER

(b) No: at $x = -8{,}500$ the distance is $\dfrac{9}{8{,}501} > 0.001$. $N = -8{,}999$ works; in general $N = 1 - \dfrac{9}{t}$. (c) From below.

SOLUTION

(a) For $x \neq 1$,

$$f(x) - 2 = \frac{2x + 7 - 2(x-1)}{x - 1} = \frac{9}{x - 1}.$$

For $x < 1$ the bottom $x - 1$ is negative, so $|f(x) - 2| = \dfrac{9}{1 - x}$.

(b) For $x < 1$, $\dfrac{9}{1-x} < t$ exactly when $1 - x > \dfrac{9}{t}$, that is, when $x < 1 - \dfrac{9}{t}$. For $t = 0.001$, $\dfrac{9}{t} = 9{,}000$, so the inputs on the left that are within 0.001 of 2 are exactly those with $x < -8{,}999$.

$N = -8{,}000$ is not far enough out: $x = -8{,}500$ is less than –8,000, but $|f(-8{,}500) - 2| = \dfrac{9}{8{,}501}$, which is more than $\dfrac{9}{9{,}000} = 0.001$.

$N = 1 - 9{,}000 = -8{,}999$ works: every $x < -8{,}999$ has $|f(x) - 2| < 0.001$. For a general tolerance, $N = 1 - \dfrac{9}{t}$ works. This N is less than 1, so every $x < N$ also has $x < 1$ and the formula of (a) applies. An N exists for every $t > 0$, so the limit is 2.

(c) From below: for $x < 1$, $f(x) - 2 = \dfrac{9}{x - 1}$ has a positive top and a negative bottom, so $f(x) < 2$.

3. Find the horizontal asymptote at each end of the graph of

$$f(x) = \frac{3x + \sqrt{36x^2 + 5}}{2x + 2}.$$

Say which end each one belongs to.

ANSWER

$y = \dfrac{9}{2}$ as $x \to \infty$ and $y = -\dfrac{3}{2}$ as $x \to -\infty$.

SOLUTION

For $x \neq 0$, factor x^2 out under the root and use $\sqrt{x^2} = |x|$:

$$\sqrt{36x^2 + 5} = |x|\sqrt{36 + \frac{5}{x^2}}.$$

As $x \to \infty$, $|x| = x$. Divide the top and the bottom by x:

$$f(x) = \frac{3 + \sqrt{36 + \frac{5}{x^2}}}{2 + \frac{2}{x}} \to \frac{3 + 6}{2} = \frac{9}{2},$$

by the basic limit and the root law (the expression under the root tends to 36 > 0).

As $x \to -\infty$, $|x| = -x$, so dividing the root by x leaves a minus sign:

$$f(x) = \frac{3 - \sqrt{36 + \frac{5}{x^2}}}{2 + \frac{2}{x}} \to \frac{3 - 6}{2} = -\frac{3}{2}.$$

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So $y = \dfrac{9}{2}$ is the asymptote on the right and $y = -\dfrac{3}{2}$ the one on the left. The root term is close to $6|x|$, which adds to $3x$ on the right and subtracts from it on the left.

4. Find each limit.

(a) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+9}-x\right)$ (b) $\lim\limits_{x\to\infty} x\left(\sqrt{x^2+9}-x\right)$

ANSWER

(a) 0 (b) $\dfrac{9}{2}$

SOLUTION

Both $\sqrt{x^2+9}$ and x grow without bound, so the difference law cannot be used. Multiply top and bottom by the conjugate $\sqrt{x^2+9}+x$:

$$\sqrt{x^2+9}-x = \frac{(x^2+9)-x^2}{\sqrt{x^2+9}+x} = \frac{9}{\sqrt{x^2+9}+x}.$$

(a) For $x>0$, divide top and bottom by x, using $\sqrt{x^2+9} = x\sqrt{1+\frac{9}{x^2}}$:

$$\frac{\frac{9}{x}}{\sqrt{1+\frac{9}{x^2}}+1} \to \frac{0}{1+1} = 0.$$

(b) The product law cannot be used on $x\cdot\left(\sqrt{x^2+9}-x\right)$ directly, because x has no limit that is a number. Multiply the rewritten difference by x and divide top and bottom by x:

$$x\left(\sqrt{x^2+9}-x\right) = \frac{9x}{\sqrt{x^2+9}+x} = \frac{9}{\sqrt{1+\frac{9}{x^2}}+1} \to \frac{9}{2}.$$

5. Let $f(x) = x\sqrt{\dfrac{x+5}{x-1}}$, which is defined for $x\le -5$ and for $x>1$.

(a) Use the two limits $m = \lim \dfrac{f(x)}{x}$ and $b = \lim\left[f(x)-mx\right]$ to find the oblique asymptote of the graph of f as $x\to\infty$.

(b) Find the oblique asymptote as $x\to -\infty$.

ANSWER

(a) $y = x+3$ (b) $y = x+3$, the same line.

SOLUTION

Write $q(x) = \dfrac{x+5}{x-1}$. Dividing its top and bottom by x gives $q(x) = \dfrac{1+\frac{5}{x}}{1-\frac{1}{x}}$, which tends to 1 as $x\to\infty$ and as $x\to-\infty$ (basic limit with the positive integer power 1).

(a) $m = \lim_{x\to\infty}\dfrac{f(x)}{x} = \lim_{x\to\infty}\sqrt{q(x)} = \sqrt{1} = 1$ by the root law, since $q(x)\to 1>0$.

Next, $f(x)-x = x\left(\sqrt{q(x)}-1\right)$. One factor grows and the other tends to 0, so rewrite with the conjugate:

$$\sqrt{q(x)}-1 = \frac{q(x)-1}{\sqrt{q(x)}+1}, \qquad q(x)-1 = \frac{(x+5)-(x-1)}{x-1} = \frac{6}{x-1}.$$

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So

$$f(x) - x = \frac{6x}{(x-1)\left(\sqrt{q(x)}+1\right)} = \frac{6}{\left(1-\frac{1}{x}\right)\left(\sqrt{q(x)}+1\right)} \to \frac{6}{1\cdot 2} = 3.$$

So $b = 3$ and the asymptote on the right is $y = x + 3$.

(b) Nothing in (a) used the sign of x: the factor x stands outside the root, so no $|x|$ appears, and every limit used holds as $x \to -\infty$ as well. So $m = 1$ and $b = 3$ again, and $y = x + 3$ is also the asymptote on the left. Compare $\sqrt{x^2 + 4x}$, where x comes out of the root as $|x|$ and the two ends get different lines.

6. Write a formula for a function f, defined for every real x, with $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = -3$. Use the limit laws to show that your function has both limits.

ANSWER

One example: $f(x) = -1 + \dfrac{2x}{\sqrt{x^2+1}}$. Other answers are possible; any correct one must be checked at both ends.

SOLUTION

A rational function cannot work, since it has the same horizontal asymptote at both ends. Build from $\dfrac{x}{\sqrt{x^2+1}}$, which is defined everywhere because $x^2 + 1 > 0$. For $x \neq 0$, $\sqrt{x^2+1} = |x|\sqrt{1+\frac{1}{x^2}}$, so

$$\frac{x}{\sqrt{x^2+1}} = \frac{x}{|x|} \cdot \frac{1}{\sqrt{1+\frac{1}{x^2}}}.$$

As $x \to \infty$, $\dfrac{x}{|x|} = 1$ and the limit is 1; as $x \to -\infty$, $\dfrac{x}{|x|} = -1$ and the limit is –1.

Now take $f(x) = c + d\,\dfrac{x}{\sqrt{x^2+1}}$ with $c + d = 1$ and $c - d = -3$, so $c = -1$ and $d = 2$:

$$f(x) = -1 + \frac{2x}{\sqrt{x^2+1}}.$$

By the sum and constant multiple laws, $\lim_{x\to\infty} f(x) = -1 + 2(1) = 1$ and $\lim_{x\to-\infty} f(x) = -1 + 2(-1) = -3$.

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