Calculus 1 Unit 1 worksheet: Discontinuities and how to remove them (answer key)

ANSWERS AT A GLANCE

1. (a) Removable at 2 (the limit there is $\frac{3}{4}$); infinite at 0 and at -2. (b) Jump at 0 of size 1. (c) Oscillating at 0.
2. (a) Removable at -4, where $\lim_{x\to -4} g(x) = -24$. Infinite at -2, with $\lim_{x\to -2^-} g(x) = -\infty$ and $\lim_{x\to -2^+} g(x) = \infty$. (b) $G(x) = \dfrac{x^2 - 4x + 16}{x + 2}$, for every real $x \neq -2$.
3. (a) Infinite at -1 ($-\infty$ from the left, ∞ from the right). Removable at 0 (the limit is 1 but $f(0) = 3$). Jump at π of size $\pi - 3$, continuous from the right and not from the left. (b) Only the one at 0: change the value to $f(0) = 1$. At -1 and at π the two one-sided limits are not equal numbers, and no value at the point changes a one-sided limit.
4. f itself has a jump of size 4 at 2 (one-sided limits 5 and 1). (a) A jump of size 4. (b) Continuous at 2. (c) A jump of size 12.
5. (a) False: the discontinuity is infinite. (b) True. (c) False: for example $f(x) = 1$ for $x > 0$, $f(x) = -1$ for $x < 0$, $f(0) = 0$ has a jump at 0, while f^2 has a removable discontinuity there.
6. Removable at 0, removed by the value 1. Infinite at $\frac{\pi}{2}$: $\lim_{x\to \pi/2^-} f(x) = \infty$ and $\lim_{x\to \pi/2^+} f(x) = -\infty$. Infinite at $-\frac{\pi}{2}$: $\lim_{x\to -\pi/2^-} f(x) = -\infty$ and $\lim_{x\to -\pi/2^+} f(x) = \infty$.

WORKED SOLUTIONS

1. Find every discontinuity of each function and classify it as removable, jump, infinite or oscillating. Give the size of every jump.

(a) $\dfrac{x^2 + 2x - 8}{x^3 - 4x}$ (b) $\dfrac{1 - \cos x}{x|x|}$ (c) $\dfrac{1}{x}\cos\dfrac{1}{x}$

ANSWER

(a) Removable at 2 (the limit there is $\frac{3}{4}$); infinite at 0 and at -2. (b) Jump at 0 of size 1. (c) Oscillating at 0.

SOLUTION

(a) The bottom is $x^3 - 4x = x(x-2)(x+2)$, so the candidates are 0, 2 and -2; the function is continuous everywhere else. The top is $(x+4)(x-2)$, so for $x \neq 0, \pm 2$ the function equals $\dfrac{x+4}{x(x+2)}$.

At 2 the simplified form is continuous, so both one-sided limits are $\dfrac{6}{2\cdot 4} = \dfrac{3}{4}$. The limit exists and the value does not: removable.

At 0 the simplified form behaves like $\dfrac{4}{2x} = \dfrac{2}{x}$: the limit from the right is ∞ and from the left $-\infty$. Infinite.

At -2 the top of the simplified form approaches 2 and x approaches -2, so the form behaves like $\dfrac{2}{-2(x+2)} = -\dfrac{1}{x+2}$: the limit from the right is $-\infty$ and from the left ∞. Infinite.

(b) The bottom is 0 only at 0, so that is the only discontinuity. For $x > 0$, $x|x| = x^2$ and the function is $\dfrac{1-\cos x}{x^2}$, which approaches $\dfrac{1}{2}$. For $x < 0$, $x|x| = -x^2$, and the function approaches $-\dfrac{1}{2}$. Both one-sided limits are numbers and they differ: a jump of size $\left|\dfrac{1}{2} - \left(-\dfrac{1}{2}\right)\right| = 1$.

(c) The only discontinuity is at 0. At $x_k = \dfrac{1}{2\pi k}$ the value is $2\pi k \cos(2\pi k) = 2\pi k$, and at $y_k = \dfrac{1}{\pi + 2\pi k}$ it is $-(\pi + 2\pi k)$, for $k = 1, 2, 3, \ldots$ Both lists of inputs approach 0 from the right, yet the values climb without bound along one and fall without bound along the other. So $\lim_{x\to 0^+}$ is not a number, not ∞ and not $-\infty$. The function is odd, so the left side behaves the same way. Neither side runs off to infinity and neither settles: oscillating, with swings that grow without bound in both directions.

Free sample from math.wiki for classroom use. Do not post. math.wiki

2. Let $g(x) = \dfrac{x^3 + 64}{x^2 + 6x + 8}$.

(a) Find and classify every discontinuity of g. At an infinite discontinuity, give both one-sided limits.

(b) Write the continuous extension of g at its removable discontinuity as one formula, and give its domain.

ANSWER

(a) Removable at -4, where $\lim_{x \to -4} g(x) = -24$. Infinite at -2, with $\lim_{x \to -2^-} g(x) = -\infty$ and $\lim_{x \to -2^+} g(x) = \infty$.

(b) $G(x) = \dfrac{x^2 - 4x + 16}{x + 2}$, for every real $x \neq -2$.

SOLUTION

(a) The bottom factors as $(x + 4)(x + 2)$, so g is continuous at every number except -4 and -2, where it is undefined but defined on both sides. The top factors as $x^3 + 64 = (x + 4)(x^2 - 4x + 16)$, so for $x \neq -4, -2$,

$$g(x) = \frac{x^2 - 4x + 16}{x + 2}$$

At -4 the simplified form is continuous (its bottom there is $-2 \neq 0$), so both one-sided limits equal $\dfrac{3 \cdot 16}{-4 - (-2)} = -24$. The limit exists and $g(-4)$ does not: removable.

At -2 the top of the simplified form approaches 28. In fact $x^2 - 4x + 16$ is positive for every x, since its discriminant is $-3 \cdot 16 < 0$. The bottom $x + 2$ approaches 0 with its own sign, so $\lim_{x \to -2^-} g(x) = -\infty$ and $\lim_{x \to -2^+} g(x) = \infty$: infinite.

(b) The extension keeps g at every $x \neq -4$ and takes the value -24 at -4. The simplified formula does both: it equals g wherever g is defined, and it gives -24 at -4. So $G(x) = \dfrac{x^2 - 4x + 16}{x + 2}$, with domain every real number except -2. No value at -2 can be added, because an infinite discontinuity cannot be removed.

3. Let

$$f(x) = \begin{cases} \dfrac{1}{x + 1}, & x < 0 \\ 3, & x = 0 \\ \cos x, & 0 < x < \pi \\ x - 4, & x \geq \pi \end{cases}$$

(a) Find and classify every discontinuity of f. For a jump, give its size and say from which side, if any, f is continuous.

(b) Which discontinuity can be removed, and how? Why can the others not be removed?

ANSWER

(a) Infinite at -1 ($-\infty$ from the left, ∞ from the right). Removable at 0 (the limit is 1 but $f(0) = 3$). Jump at π of size $\pi - 3$, continuous from the right and not from the left.

(b) Only the one at 0: change the value to $f(0) = 1$. At -1 and at π the two one-sided limits are not equal numbers, and no value at the point changes a one-sided limit.

SOLUTION

Free sample from math.wiki for classroom use. Do not post. math.wiki

(a) Inside each open interval where one rule holds, the rule is continuous: $\dfrac{1}{x+1}$ is continuous for $x < 0$ except at -1, and $\cos x$ and $x - 4$ are continuous. So the candidates are -1, 0 and π.

At -1: f is undefined, and $\dfrac{1}{x+1}$ tends to $-\infty$ from the left and ∞ from the right. Infinite.

At 0: $\lim_{x\to 0^-} f(x) = \dfrac{1}{0+1} = 1$ and $\lim_{x\to 0^+} f(x) = \cos 0 = 1$. The limit is 1, but $f(0) = 3$: removable.

At π: $\lim_{x\to \pi^-} f(x) = \cos \pi = -1$ and $\lim_{x\to \pi^+} f(x) = \pi - 4$. Both are numbers and they differ, since $\pi - 4 \approx -0.86$. A jump of size $|(\pi - 4) - (-1)| = \pi - 3$. The value $f(\pi) = \pi - 4$ matches the right-hand limit, so f is continuous from the right at π and not from the left.

(b) Redefining $f(0) = 1$ makes the limit at 0 equal the value, so f becomes continuous at 0, and 1 is the only value that does this. Changing the value at a point leaves both one-sided limits alone. At -1 one side is infinite and at π the sides differ, so $\lim f(x)$ fails to exist as a number there whatever value is chosen.

4. Let $f(x) = 7 - x$ for $x < 2$ and $f(x) = x^2 - 3$ for $x \geq 2$. Classify each function below at 2. For a jump, give its size.

(a) $f(x) + x^3$ (b) $(x^2 - 2x)f(x)$ (c) $(x^2 - 7)f(x)$

ANSWER

f itself has a jump of size 4 at 2 (one-sided limits 5 and 1). (a) A jump of size 4. (b) Continuous at 2. (c) A jump of size 12.

SOLUTION

From the rules, $\lim_{x\to 2^-} f(x) = 7 - 2 = 5$ and $\lim_{x\to 2^+} f(x) = 4 - 3 = 1$, with $f(2) = 1$: a jump of size 4. Each multiplier and added term below is a polynomial, so it is continuous at 2, and the results on sums and products apply.

(a) One-sided limits $5 + 8 = 13$ and $1 + 8 = 9$: adding a continuous function keeps the jump, of size 4.

(b) The factor $x^2 - 2x = x(x - 2)$ has value 0 at 2. One-sided limits $0 \cdot 5 = 0$ and $0 \cdot 1 = 0$, and the value is $0 \cdot f(2) = 0$. Continuous at 2.

(c) The factor $g(x) = x^2 - 7$ has value -3 at 2. One-sided limits $-3 \cdot 5 = -15$ and $-3 \cdot 1 = -3$: a jump of size $|-3| \cdot 4 = 12$. The jump changes direction (it now steps up), but its size is still $|g(2)|$ times the size of the jump of f.

5. Decide whether each statement is true for every function f and every number a. Prove a true one; give a counterexample to a false one.

(a) If $\lim_{x\to a^-} f(x) = 2$ and $\lim_{x\to a^+} f(x) = \infty$, then f has a jump at a.

(b) If f has a removable discontinuity at a, then $3f + 1$ has a removable discontinuity at a.

(c) If f has a jump at a, then f^2 has a jump at a.

ANSWER

(a) False: the discontinuity is infinite. (b) True. (c) False: for example $f(x) = 1$ for $x > 0$, $f(x) = -1$ for $x < 0$, $f(0) = 0$ has a jump at 0, while f^2 has a removable discontinuity there.

SOLUTION

(a) False. A jump needs both one-sided limits to be numbers. Here one side runs off to ∞, so the discontinuity is infinite, whatever happens on the other side. For example $f(x) = 2$ for $x < 0$ and $f(x) = \dfrac{1}{x}$ for $x > 0$.

(b) True. Let $\lim_{x\to a} f(x) = L$, with $f(a)$ undefined or $f(a) \neq L$. By the limit laws, $\lim_{x\to a} \big(3f(x) + 1\big) = 3L + 1$, so the limit exists. If $f(a)$ is undefined, so is $3f(a) + 1$. If $f(a) \neq L$, then $3f(a) + 1 \neq 3L + 1$.

Free sample from math.wiki for classroom use. Do not post. math.wiki

Either way $3f + 1$ is not continuous at a while its limit exists: removable.

(c) False. Take $f(x) = 1$ for $x > 0$, $f(x) = -1$ for $x < 0$ and $f(0) = 0$. Its one-sided limits at 0 are −1 and 1: a jump of size 2. But $f(x)^2 = 1$ for every $x \neq 0$, so both one-sided limits of f^2 are 1, while $f(0)^2 = 0$. That is a removable discontinuity, not a jump.

6. Let $f(x) = \dfrac{\tan x}{x}$ on the interval $(-\pi, \pi)$. Find and classify every discontinuity of f in this interval. Give the one-sided limits at each infinite discontinuity, and the value that removes any removable one.

ANSWER

Removable at 0, removed by the value 1. Infinite at $\dfrac{\pi}{2}$: $\lim_{x \to \pi/2^-} f(x) = \infty$ and $\lim_{x \to \pi/2^+} f(x) = -\infty$. Infinite at $-\dfrac{\pi}{2}$: $\lim_{x \to -\pi/2^-} f(x) = -\infty$ and $\lim_{x \to -\pi/2^+} f(x) = \infty$.

SOLUTION

On $(-\pi, \pi)$ the function is undefined where $x = 0$ and where $\tan x$ is undefined, at $\pm\dfrac{\pi}{2}$. Everywhere else it is a quotient of continuous functions with a nonzero bottom, so it is continuous. Each of the three numbers has the function defined on both sides.

At 0: $\dfrac{\tan x}{x} = \dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x}$, and the two factors approach 1 and 1. The limit is 1 and $f(0)$ is undefined: removable, and setting the value 1 at 0 removes it.

At $\dfrac{\pi}{2}$: $\tan x \to \infty$ from the left and $-\infty$ from the right, while x approaches the positive number $\dfrac{\pi}{2}$. Dividing by a number near $\dfrac{\pi}{2}$ keeps the signs: ∞ from the left, $-\infty$ from the right. Infinite.

At $-\dfrac{\pi}{2}$: $\tan x \to \infty$ from the left and $-\infty$ from the right, but now x is near the negative number $-\dfrac{\pi}{2}$, which flips the signs: $-\infty$ from the left, ∞ from the right. Infinite.

Free sample from math.wiki for classroom use. Do not post. math.wiki