Calculus 1 Unit 1 worksheet: The squeeze theorem and the limit of sine x over x (answer key)

ANSWERS AT A GLANCE

1. (a) 0 (b) 0 (c) 5
2. (a) $\frac{15}{2}$ (b) $\frac{8}{9}$ (c) 0
3. (a) 7 (b) 0
4. $\frac{13}{5}$
5. $\frac{\pi}{3}$
6. (b) $P(4) = 24\sqrt{2}$ cm, $P(6) = 36$ cm. (c) 12π cm, the circumference of the circle.
7. (a) Only $\lim_{x \to 0^+} f(x) = 0$; the bounds say nothing for $x < 0$. (b) For example $f(x) = x$ for $x > 0$ and $f(x) = 1$ for $x \le 0$: the right-hand limit is 0 and the left-hand limit is 1.

WORKED SOLUTIONS

1. Use the squeeze theorem to find each limit. Name the two walls you use.

(a) $\lim_{x \to 2} (x-2)^2 \cos \dfrac{1}{x-2}$ (b) $\lim_{x \to 0} \dfrac{x^2}{3 + \sin \frac{1}{x}}$ (c) $\lim_{x \to 0} \left(5 + x \cos \dfrac{4}{x}\right)$

ANSWER

(a) 0 (b) 0 (c) 5

SOLUTION

(a) For $x \neq 2$, $-1 \le \cos \dfrac{1}{x-2} \le 1$. Multiply by $(x-2)^2$, which is positive for $x \neq 2$:

$$-(x-2)^2 \le (x-2)^2 \cos \frac{1}{x-2} \le (x-2)^2$$

Both walls are polynomials with limit 0 at 2, so the limit is 0.

(b) Since $-1 \le \sin \dfrac{1}{x} \le 1$, the bottom satisfies $2 \le 3 + \sin \dfrac{1}{x} \le 4$, so it is positive and its reciprocal lies between $\frac{1}{4}$ and $\frac{1}{2}$. Multiply by $x^2 \ge 0$:

$$\frac{x^2}{4} \le \frac{x^2}{3 + \sin \frac{1}{x}} \le \frac{x^2}{2} \qquad (x \neq 0)$$

Both walls tend to 0, so the limit is 0.

(c) For $x \neq 0$, $\left| x \cos \dfrac{4}{x} \right| \le |x|$, so $-|x| \le x \cos \dfrac{4}{x} \le |x|$. Both walls tend to 0, so $x \cos \dfrac{4}{x} \to 0$ by the squeeze theorem. (The walls are $\pm|x|$, not $\pm x$: for negative x the wall x lies below $-x$.) By the sum law the limit is 5 + 0 = 5.

2. Find each limit. Rewrite each one with factors of the form $\dfrac{\sin u}{u}$ or $\dfrac{\tan u}{u}$, and say what u is.

(a) $\lim_{x \to 0} \dfrac{\sin 3x \, \sin 5x}{x \tan 2x}$ (b) $\lim_{x \to 0} \dfrac{1 - \cos 4x}{\sin^2 3x}$ (c) $\lim_{x \to 0^+} \dfrac{\tan x}{\sqrt{x}}$

ANSWER

(a) $\frac{15}{2}$ (b) $\frac{8}{9}$ (c) 0

SOLUTION

(a) Give each input its own denominator and put back what was borrowed:

$$\frac{\sin 3x \, \sin 5x}{x \tan 2x} = \frac{\sin 3x}{3x} \cdot \frac{\sin 5x}{5x} \cdot \frac{2x}{\tan 2x} \cdot \frac{15x^2}{2x^2}$$

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The first two factors tend to 1, the third to $\frac{1}{1} = 1$, and the last is $\frac{15}{2}$ for $x \neq 0$. The limit is $\frac{15}{2}$.

(b) Multiply top and bottom by $1 + \cos 4x$, which is positive near 0, and use $1 - \cos^2 4x = \sin^2 4x$:

$$\frac{1 - \cos 4x}{\sin^2 3x} = \left(\frac{\sin 4x}{4x}\right)^2 \cdot \left(\frac{3x}{\sin 3x}\right)^2 \cdot \frac{16x^2}{9x^2} \cdot \frac{1}{1 + \cos 4x}$$

The factors tend to 1, 1, $\frac{16}{9}$ and $\frac{1}{2}$, so the limit is $\frac{16}{9} \cdot \frac{1}{2} = \frac{8}{9}$.

(c) For $x > 0$, $\frac{\tan x}{\sqrt{x}} = \frac{\tan x}{x} \cdot \sqrt{x}$. The first factor tends to 1 and $\sqrt{x} \to 0$ as $x \to 0^+$, so the limit is $1 \cdot 0 = 0$.

3. A function f satisfies $|f(x) - 7| \leq 3(x-2)^2$ for every x.

(a) Find $\lim_{x \to 2} f(x)$.

(b) Find $\lim\limits_{x \to 2} \dfrac{f(x) - 7}{x - 2}$.

ANSWER

(a) 7 (b) 0

SOLUTION

(a) The inequality says $7 - 3(x-2)^2 \leq f(x) \leq 7 + 3(x-2)^2$. Both walls are polynomials that tend to 7 as $x \to 2$, so by the squeeze theorem $\lim_{x \to 2} f(x) = 7$.

(b) The quotient law cannot be used, since the bottom tends to 0. For $x \neq 2$, divide the given inequality by the positive number $|x - 2|$:

$$\left|\frac{f(x) - 7}{x - 2}\right| \leq 3|x - 2|, \quad \text{so} \quad -3|x-2| \leq \frac{f(x) - 7}{x - 2} \leq 3|x - 2|$$

Both walls tend to 0 as $x \to 2$, so by the squeeze theorem the limit is 0.

4. Find $\lim\limits_{x \to 0} \dfrac{5x + \sin 8x}{\tan 5x}$.

ANSWER

$\frac{13}{5}$

SOLUTION

Substituting gives $\frac{0}{0}$. For $x \neq 0$ near 0, divide top and bottom by x:

$$\frac{5x + \sin 8x}{\tan 5x} = \frac{5 + 8 \cdot \frac{\sin 8x}{8x}}{5 \cdot \frac{\tan 5x}{5x}}$$

By rescaling, $\frac{\sin 8x}{8x} \to 1$ and $\frac{\tan 5x}{5x} \to 1$. The bottom tends to $5 \neq 0$, so the sum, constant multiple and quotient laws give

$$\lim_{x \to 0} \frac{5x + \sin 8x}{\tan 5x} = \frac{5 + 8}{5} = \frac{13}{5}$$

5. Find $\lim\limits_{x \to 3} \dfrac{\sin 2\pi x}{x^2 - 9}$. (Hint: let $u = x - 3$.)

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ANSWER

$\dfrac{\pi}{3}$

SOLUTION

Substituting gives $\dfrac{\sin 6\pi}{0} = \dfrac{0}{0}$. Let $u = x - 3$, so $u \to 0$ and $u \neq 0$ as $x \to 3$. Then $2\pi x = 2\pi u + 6\pi$, and by the sum formula for sine,

$$\sin 2\pi x = \sin 2\pi u \cos 6\pi + \cos 2\pi u \sin 6\pi = \sin 2\pi u$$

since $\cos 6\pi = 1$ and $\sin 6\pi = 0$. The bottom factors as $x^2 - 9 = (x-3)(x+3) = u(x+3)$, so

$$\frac{\sin 2\pi x}{x^2 - 9} = \frac{\sin 2\pi u}{2\pi u} \cdot \frac{2\pi}{x+3}$$

By rescaling the first fraction tends to 1, and $x + 3 \to 6$. The limit is

$$\lim_{x \to 3} \frac{\sin 2\pi x}{x^2 - 9} = \frac{\pi}{3}$$

6. Priya draws a regular polygon with n sides inside a circle of radius 6 cm, with every corner on the circle.

(a) Show that the perimeter of the polygon is $P(n) = 12n \sin \dfrac{\pi}{n}$ cm.

(b) Find $P(4)$ and $P(6)$ exactly.

(c) Find $\lim_{n \to \infty} P(n)$. What length is the perimeter approaching?

ANSWER

(b) $P(4) = 24\sqrt{2}$ cm, $P(6) = 36$ cm. (c) 12π cm, the circumference of the circle.

SOLUTION

(a) Join the center to the corners. Each side subtends a central angle of $\dfrac{2\pi}{n}$. The segment from the center to the midpoint of a side is perpendicular to it and halves that angle, so half a side is $6 \sin \dfrac{\pi}{n}$ and a side is $12 \sin \dfrac{\pi}{n}$. The n sides give $P(n) = 12n \sin \dfrac{\pi}{n}$ cm.

(b) $P(4) = 48 \sin \dfrac{\pi}{4} = 48 \cdot \dfrac{\sqrt{2}}{2} = 24\sqrt{2}$ cm, about 33.94 cm. $P(6) = 72 \sin \dfrac{\pi}{6} = 72 \cdot \dfrac{1}{2} = 36$ cm.

(c) Let $u = \dfrac{\pi}{n}$. Then $n = \dfrac{\pi}{u}$, and $u \to 0^+$ with $u \neq 0$ as $n \to \infty$:

$$P(n) = 12 \cdot \frac{\pi}{u} \sin u = 12\pi \cdot \frac{\sin u}{u}$$

Since $\dfrac{\sin u}{u} \to 1$, $\lim_{n \to \infty} P(n) = 12\pi$ cm, about 37.70 cm. That is the circumference $2\pi \cdot 6$: the polygons hug the circle more and more closely.

7. All that Mateo knows about a function f is that $0 \leq f(x) \leq x$ for $0 < x < 1$. He concludes that $\lim_{x \to 0} f(x) = 0$.

(a) What can Mateo correctly conclude from his bounds, and why is his conclusion too strong?

(b) Give a function f that satisfies his bounds but has no limit at 0, and show that it has none.

ANSWER

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(a) Only $\lim_{x \to 0^+} f(x) = 0$; the bounds say nothing for $x < 0$. (b) For example $f(x) = x$ for $x > 0$ and $f(x) = 1$ for $x \leq 0$: the right-hand limit is 0 and the left-hand limit is 1.

SOLUTION

(a) The walls 0 and x both tend to 0 as $x \to 0^+$, and the trap holds for every x in $(0, 1)$, so the one-sided squeeze theorem gives $\lim_{x \to 0^+} f(x) = 0$. A two-sided limit needs the trap on an open interval around 0, on both sides, and nothing is known about $f(x)$ for $x < 0$.

(b) Let $f(x) = x$ for $x > 0$ and $f(x) = 1$ for $x \leq 0$. For $0 < x < 1$, $0 \leq x \leq x$, so the bounds hold. But $\lim_{x \to 0^+} f(x) = 0$ while $\lim_{x \to 0^-} f(x) = 1$. The one-sided limits differ, so $\lim_{x \to 0} f(x)$ does not exist.

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