Calculus 1 Unit 1 worksheet: Limits by algebra: factoring and rationalizing (answer key)

ANSWERS AT A GLANCE

1. (a) $\frac{4}{5}$ (b) -7 (c) $\frac{27}{4}$ (d) $-\frac{27}{5}$ (e) 32 (f) 0
2. (a) $\frac{1}{32}$ (b) 2 (c) $\frac{1}{4}$ (d) $-\frac{1}{18}$
3. -9
4. (a) $\frac{1}{8}$ (b) -1
5. $\frac{8}{3}$. A root appears on the top and on the bottom, so the factor $x-3$ comes out of each only through its own conjugate.
6. (a) The new numerator $x+1$ is $-1 \neq 0$ at -2, so $x+2$ is not a factor of it; the form is nonzero over zero, not $\frac{0}{0}$. (b) The limit does not exist.

WORKED SOLUTIONS

1. Evaluate each limit. Substitute first; when you get $\frac{0}{0}$, factor, cancel and substitute again.

(a) $\lim_{x\to 5}\dfrac{x^2-2x-15}{x^2-25}$

(b) $\lim_{x\to -2}\dfrac{3x^2+5x-2}{x^2+5x+6}$

(c) $\lim_{x\to 3}\dfrac{x^3-27}{x^2-2x-3}$

(d) $\lim_{x\to -3}\dfrac{x^3+27}{x^2+x-6}$

(e) $\lim_{x\to 2}\dfrac{x^4-16}{x^2-3x+2}$

(f) $\lim_{x\to 2}\dfrac{x^3-2x^2-4x+8}{x^3-4x}$

ANSWER

(a) $\frac{4}{5}$ (b) -7 (c) $\frac{27}{4}$ (d) $-\frac{27}{5}$ (e) 32 (f) 0

SOLUTION

Every one gives $\frac{0}{0}$ at its target, so by the factor theorem the top and the bottom share the factor $x-a$.

(a) $\dfrac{(x-5)(x+3)}{(x-5)(x+5)}=\dfrac{x+3}{x+5}$ for $x\neq 5$, so the limit is $\dfrac{8}{10}=\dfrac{4}{5}$.

(b) $3x^2+5x-2=(3x-1)(x+2)$ and $x^2+5x+6=(x+2)(x+3)$. For $x\neq -2$ the quotient is $\dfrac{3x-1}{x+3}$, so the limit is $\dfrac{-7}{1}=-7$.

(c) Difference of cubes on top: $\dfrac{(x-3)(x^2+3x+9)}{(x-3)(x+1)}=\dfrac{x^2+3x+9}{x+1}$ for $x\neq 3$, so the limit is $\dfrac{9+9+9}{4}=\dfrac{27}{4}$.

(d) Sum of cubes on top: $\dfrac{(x+3)(x^2-3x+9)}{(x+3)(x-2)}=\dfrac{x^2-3x+9}{x-2}$ for $x\neq -3$, so the limit is $\dfrac{9+9+9}{-5}=-\dfrac{27}{5}$.

(e) $x^4-16=(x^2-4)(x^2+4)=(x-2)(x+2)(x^2+4)$ and $x^2-3x+2=(x-1)(x-2)$. For $x\neq 2$ the quotient is $\dfrac{(x+2)(x^2+4)}{x-1}$, so the limit is $\dfrac{4\cdot 8}{1}=32$.

(f) Group the top: $x^2(x-2)-4(x-2)=(x-2)(x^2-4)=(x-2)^2(x+2)$. The bottom is $x(x-2)(x+2)$. For $x\neq 2$ (and $x\neq -2$) the quotient is $\dfrac{x-2}{x}$, whose bottom is $2\neq 0$ at $x=2$, so the limit is $\dfrac{0}{2}=0$.

2. Evaluate each limit. Multiply by a conjugate, and factor what is left if you need to.

(a) $\lim_{x\to 4}\dfrac{\sqrt{x}-2}{x^2-16}$

(b) $\lim_{x\to -1}\dfrac{x+1}{\sqrt{2x+6}-2}$

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(c) $\lim_{x\to 5}\dfrac{\sqrt{3x+1}-\sqrt{x+11}}{x-5}$ (d) $\lim_{x\to 3}\dfrac{3-\sqrt{x+6}}{x^2-3x}$

ANSWER

(a) $\dfrac{1}{32}$ (b) 2 (c) $\dfrac{1}{4}$ (d) $-\dfrac{1}{18}$

SOLUTION

(a) The form is $\dfrac{0}{0}$. Multiply top and bottom by $\sqrt{x}+2$: the top becomes $x-4$, and the bottom x^2-16 still has to be factored as $(x-4)(x+4)$. For $x\neq 4$ the quotient is $\dfrac{1}{(x+4)(\sqrt{x}+2)}$, so the limit is $\dfrac{1}{8\cdot 4}=\dfrac{1}{32}$.

(b) At $x=-1$ the bottom is $\sqrt{4}-2=0$. Multiply top and bottom by $\sqrt{2x+6}+2$: the bottom becomes $(2x+6)-4=2(x+1)$. For $x\neq -1$ the quotient is $\dfrac{\sqrt{2x+6}+2}{2}$, so the limit is $\dfrac{2+2}{2}=2$.

(c) At $x=5$ both roots are 4. Multiply by $\sqrt{3x+1}+\sqrt{x+11}$: the top becomes $(3x+1)-(x+11)=2x-10=2(x-5)$. For $x\neq 5$ the quotient is $\dfrac{2}{\sqrt{3x+1}+\sqrt{x+11}}$, so the limit is $\dfrac{2}{4+4}=\dfrac{1}{4}$.

(d) At $x=3$ the top is 3 − 3 = 0 and the bottom is 0. Multiply by $3+\sqrt{x+6}$: the top becomes $9-(x+6)=3-x=-(x-3)$, and the bottom is $x(x-3)(3+\sqrt{x+6})$. For $x\neq 3$ the quotient is $\dfrac{-1}{x(3+\sqrt{x+6})}$, so the limit is $\dfrac{-1}{3\cdot 6}=-\dfrac{1}{18}$.

3. Expand the numerator, cancel the factor h, and evaluate the limit.

$$\lim_{h\to 0}\frac{(3+h)^2(3-3h)-27}{h}$$

ANSWER

-9

SOLUTION

At $h=0$ the numerator is $9\cdot 3-27=0$ and the bottom is 0, so the form is $\dfrac{0}{0}$, but no factor h shows yet. Expand: $(3+h)^2=9+6h+h^2$, and multiplying by $3-3h$ and collecting powers of h gives

$$(3+h)^2(3-3h)=27-9h-15h^2-3h^3.$$

The constants cancel, as they must, so the numerator is

$$-9h-15h^2-3h^3=h\big(-9-15h-3h^2\big).$$

For $h\neq 0$ the quotient equals the polynomial $-9-15h-3h^2$, so by the agreement principle and direct substitution the limit is −9.

4. Combine the fractions in the numerator, then evaluate each limit.

(a) $\lim_{x\to 2}\dfrac{\frac{x}{x+2}-\frac{1}{2}}{x-2}$ (b) $\lim_{x\to 1}\dfrac{\frac{2}{x^2+1}-1}{x-1}$

ANSWER

(a) $\dfrac{1}{8}$ (b) -1

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SOLUTION

(a) At $x = 2$ the top is $\dfrac{2}{4} - \dfrac{1}{2} = 0$ and the bottom is 0. Over the common denominator $2(x+2)$,

$$\frac{x}{x+2} - \frac{1}{2} = \frac{2x - (x+2)}{2(x+2)} = \frac{x-2}{2(x+2)}.$$

Dividing by $x - 2$, for $x \neq 2$, leaves $\dfrac{1}{2(x+2)}$, whose bottom is 8 at $x = 2$. The limit is $\dfrac{1}{8}$.

(b) At $x = 1$ the top is $\dfrac{2}{2} - 1 = 0$ and the bottom is 0. Over the common denominator $x^2 + 1$,

$$\frac{2}{x^2+1} - 1 = \frac{2 - (x^2+1)}{x^2+1} = \frac{1 - x^2}{x^2+1} = \frac{-(x-1)(x+1)}{x^2+1}.$$

Dividing by $x - 1$, for $x \neq 1$, leaves $\dfrac{-(x+1)}{x^2+1}$, so the limit is $\dfrac{-2}{2} = -1$.

5. Evaluate the limit below, and explain why multiplying by one conjugate is not enough.

$$\lim_{x \to 3} \frac{\sqrt{2x+3} - 3}{\sqrt{x+13} - 4}$$

ANSWER

$\dfrac{8}{3}$. A root appears on the top and on the bottom, so the factor $x - 3$ comes out of each only through its own conjugate.

SOLUTION

At $x = 3$ the top is $\sqrt{9} - 3 = 0$ and the bottom is $\sqrt{16} - 4 = 0$. One conjugate clears the root from one line only: multiplying by $\sqrt{2x+3} + 3$ turns the top into $2x - 6$ but leaves $\sqrt{x+13} - 4$ in the bottom, which shows no factor $x - 3$. So multiply top and bottom by both conjugates, $\sqrt{2x+3} + 3$ and $\sqrt{x+13} + 4$, each positive near 3:

$$\frac{\big((2x+3) - 9\big)(\sqrt{x+13} + 4)}{\big((x+13) - 16\big)(\sqrt{2x+3} + 3)} = \frac{2(x-3)(\sqrt{x+13} + 4)}{(x-3)(\sqrt{2x+3} + 3)}.$$

For $x \neq 3$ this is $\dfrac{2(\sqrt{x+13} + 4)}{\sqrt{2x+3} + 3}$, a quotient whose bottom is $6 \neq 0$ at 3, so the limit is $\dfrac{2 \cdot 8}{6} = \dfrac{8}{3}$.

6. Ava evaluates $\lim_{x \to -2} \dfrac{x^2 + 3x + 2}{x^2 + 4x + 4}$. She factors, cancels one factor $x + 2$ to get $\dfrac{x+1}{x+2}$, substitutes, and gets $\dfrac{-1}{0}$. She says: "There is still a zero in the bottom, so I should cancel again."

(a) Explain why there is nothing more to cancel.

(b) Find the limit, or explain why it does not exist.

ANSWER

(a) The new numerator $x + 1$ is $-1 \neq 0$ at -2, so $x + 2$ is not a factor of it; the form is nonzero over zero, not $\dfrac{0}{0}$. (b) The limit does not exist.

SOLUTION

(a) A second cancellation needs a second factor $x + 2$ on top. By the factor theorem, $x + 2$ is a factor of $x + 1$ only if $x + 1$ is 0 at $x = -2$, and it is -1. The original top is $(x+1)(x+2)$ and the bottom is $(x +$

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$2)^2$: the bottom has the factor twice, the top only once. A 0 in the bottom calls for canceling only when the top is 0 too.

(b) For $x \neq -2$ the quotient equals $\dfrac{x+1}{x+2}$. Its top approaches -1 while its bottom approaches 0, so its size grows without bound and it settles at no number: the limit does not exist. Outputs show it: at $x = -2.01$ the quotient is $\dfrac{-1.01}{-0.01} = 101$, and at $x = -1.99$ it is $\dfrac{-0.99}{0.01} = -99$.

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